13th National Conference on Mechanisms and Machines (NaCoMM07),
IISc, Bangalore, India. December 12-13, 2007

NaCoMM-2007-091

Classical screw theory: a fresh look using the dual eigenproblem approach

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Abstract

This paper presents a novel algebraic formulation of the central problem of screw theory, namely the determination of the principal screws of a given system. Using the algebra of dual numbers, it shows that the principal screws can be determined via the solution of a generalised eigenproblem of two real, symmetric matrices. This approach allows the study of the principal screws of the general screw systems associated with a manipulator of arbitrary geometry in terms of closed-form expressions of its architecture and configuration parameters. The formulation is illustrated with examples of practical manipulators.
Keywords: Screw theory, Principal screws, Eigenproblem, and Dual number

1 Introduction

The theory of screws has been used to analyse finite and instantaneous motions of rigid bodies over the past few centuries. One of the major applications of screw theory has been in describing the instantaneous motion of a rigid body in terms of the *principal screws*, which form a canonical basis of the motion space. The elements of screw theory emerged from the works of Giulio Mozzi [1] (1763 A.D.), Cauchy, and Chasles [2] (1830 A.D.). However, in 1900, Sir Robert Ball [3] established formally the theory of screws and applied it to the analysis of rigid-body motions of multiple degrees-of-freedom. The fundamental concepts of classical screw theory, including the principal screws of a screw system, the cylindroid, the pitch hyperboloid, and the reciprocity of screws were introduced in his treatise. In 1976, Hunt [4] rejuvenated screw theory from geometric considerations and applied it to the analysis and synthesis of mechanisms.

The determination of the principal screws of a given system has attracted a significant amount of research ever since Hunt's contribution. Interestingly, in spite of the long history of classical screw theory and the number of contributions in the area of principal screws, there seems to be a lack of variety in the reported works. Researchers have mostly retraced the geometric construction of the principal screws in one way or the other. For example, Zhang and Xu [5] have reconstructed the pitch hyperboloid algebraically for a general three-system from the three cylindroids corresponding to the distinct pairs of input screws. They have computed the principal pitches from the *normal form* of the hyperboloid. However, their approach involves solutions of (up to 9) simultaneous linear equations. Therefore it is procedural in nature and generates only numerical solutions. Fang and Huang [6] have used the *planar representation* of a third-order screw system and used the condition of degeneracy of the conic-sections in this plane to identify the principal screws. Huang and Wang [7] used the above formulation in the context of a 3-RPS pyramid manipulator and identified numerically its principal screws at different configurations. Huang, Wang and Fang [8] have studied the 3-RPS manipulator proposed by Lee and Shah [9], and termed it a *deficient-rank* manipulator as the associated three-system is found to belong to the fourth or the seventh special systems discussed in [4]. Huang, Li and Zuo [10] have combined the concepts of reciprocal screws, planar degeneration etc. described above to analyse the feasible motions of a special 3-UPU manipulator.

In [11], Bandyopadhyay and Ghosal have proposed a new approach based on the eigenproblem of a symmetric dual matrix. The complete solution of the dual eigenproblem requires the solution of a *generalised eigenproblem* involving two real symmetric matrices (namely the real and dual parts of the dual matrix), and leads to the set of principal screws. In this paper, we apply this method to study the principal screws of general two-, three-, four-, and five-systems. We show that for two-, and three-systems, the generalised eigenproblem leads to a quadratic and a cubic equation respectively. Therefore a closed-form exact solution is always obtainable. We derive the expressions for the principal screws and their pitches in these cases. For the sake of completeness, we derive the classical results related to the cylindroid and the hyperboloid within our formulation, and interpret some of the special screw systems algebraically. We also present the analysis of the four-, and five-systems, which differs from the analysis of the previous two. The novelty in our work is that unlike the other formulations, it does not require the computation of the reciprocal systems explicitly to start with.

The paper is organised as follows: in section 2, we present a brief description of the mathematical formulation of our approach for the general two-, three-systems. In section 3, some closed-from results pertaining to these systems are presented. The four-, and five-system are discussed in section 4. In section 5, we present two examples to illustrate our formulation. Finally, we present the conclusions in section 6.

2 Mathematical Development

In this section, we describe the formulation very briefly. We follow the same notations and formulation presented in [11], therefore the reader is directed to [11] for the details. For an introduction to dual numbers and their use in kinematics, the reader may please refer to [12, 13].

The forward velocity equation of a n-degrees-of-freedom manipulator may be viewed as a dual-valued linear map of the joint rate vector $\dot{\boldsymbol{\theta}}$:

$$\hat{\boldsymbol{\mathcal{V}}} = \hat{\boldsymbol{J}}\dot{\boldsymbol{\theta}}, \qquad \dot{\boldsymbol{\theta}} \in \mathbb{R}^n \tag{1}$$

where $\hat{\boldsymbol{J}}$ is the *dual Jacobian matrix*, $\hat{\boldsymbol{\mathcal{V}}} = \boldsymbol{\omega} + \epsilon\boldsymbol{v}$, $\boldsymbol{\omega} = \boldsymbol{J_\omega}\dot{\boldsymbol{\theta}}$, $\boldsymbol{v} = \boldsymbol{J_v}\dot{\boldsymbol{\theta}}$ and therefore $\hat{\boldsymbol{J}} = \boldsymbol{J_\omega} + \epsilon\boldsymbol{J_v}$. To study the distributions of twists, we obtain the extremal values of the dual norm of $\hat{\boldsymbol{\mathcal{V}}}$, i.e., $\|\hat{\boldsymbol{\mathcal{V}}}\|_d$, under the constraint $\|\dot{\boldsymbol{\theta}}\| = 1$. This results in the following dual eigenproblem [11]:

$$\hat{\boldsymbol{g}}\dot{\boldsymbol{\theta}} = \hat{\lambda}\dot{\boldsymbol{\theta}} \tag{2}$$

where $\hat{\boldsymbol{g}} = \hat{\boldsymbol{J}}^T\hat{\boldsymbol{J}} = \boldsymbol{g} + \epsilon\boldsymbol{g}_0$, and $\boldsymbol{g} = \boldsymbol{J}_{\boldsymbol{\omega}}^T\boldsymbol{J_\omega}$, $\boldsymbol{g}_0 = \boldsymbol{J}_{\boldsymbol{\omega}}^T\boldsymbol{J_v} + \boldsymbol{J}_{\boldsymbol{v}}^T\boldsymbol{J_\omega}$. The dual eigenproblem above is equivalent to the following pair of real eigenproblems:

$$\boldsymbol{g}\dot{\boldsymbol{\theta}} = \lambda\dot{\boldsymbol{\theta}}, \qquad \boldsymbol{g}_0\dot{\boldsymbol{\theta}} = \lambda_0\dot{\boldsymbol{\theta}} \tag{3}$$

The two eigenproblems in Eq. (3) are consistent *iff* the matrices $\boldsymbol{g}$ and $\boldsymbol{g}_0$ share the eigenvector $\dot{\boldsymbol{\theta}}$. From linear algebra, the condition implies that

$$\boldsymbol{g}\boldsymbol{g}_0 = \boldsymbol{g}_0\boldsymbol{g} \Leftrightarrow [\boldsymbol{g}, \boldsymbol{g}_0] = \boldsymbol{0} \tag{4}$$

In general, this condition is not satisfied automatically. However, if $\boldsymbol{g}$ is positive definite, there exists a transformation $\boldsymbol{T}$ of $\mathbb{R}^n$, which reduces $\boldsymbol{g}$ and $\boldsymbol{g}_0$ to such forms that they commute. The matrix $\boldsymbol{g}$ can be positive definite only if n equals 1, 2, or 3. Neglecting the trivial case of one-DOF motion, this refers to the general two-, and three-systems. A three-step method to obtain $\boldsymbol{T}$ and their geometric significance are described in [11]. The same result can also be obtained by direct algebraic manipulation as follows. Since $\lambda \neq 0$ under the assumptions, we can write from Eq. (3):

$$\begin{aligned} \boldsymbol{g}_0\dot{\boldsymbol{\theta}} &= \frac{\lambda_0}{\lambda}\boldsymbol{g}\dot{\boldsymbol{\theta}} \\ \Rightarrow \left(\boldsymbol{g}_0 - \frac{\lambda_0}{\lambda}\boldsymbol{g}\right)\dot{\boldsymbol{\theta}} &= \boldsymbol{0}, \quad \lambda \neq 0 \end{aligned} \tag{5}$$

The above equation represents a *generalised eigenvalue problem* of $\boldsymbol{g}_0$ with respect to $\boldsymbol{g}$. It has been shown in [11] that the generalised eigenvalues, $\frac{\lambda_0}{\lambda}$, are double the principal pitches (denoted by h^h). Denoting the ith generalised eigenvector by $\dot{\boldsymbol{\theta}}_i^h$, we can rewrite Eq. (5) as:

$$\left(\boldsymbol{g}_0 - 2h_i^h\boldsymbol{g}\right)\dot{\boldsymbol{\theta}}_i^h = \boldsymbol{0} \tag{6}$$

The *generalised eigenvalues*, h_i^h, can be obtained from the generalised characteristic equation:

$$\det\left(\boldsymbol{g}_0 - 2h_i^h\boldsymbol{g}\right) = 0 \tag{7}$$

The eigenvectors form the columns of the required transformation $\boldsymbol{T}$. The principal screws are obtained from them in two steps. First, we obtain the principal twists by mapping the eigenvectors $\dot{\boldsymbol{\theta}}_i^h$ by $\hat{\boldsymbol{J}}$ (see Eq. (1)):

$$\hat{\boldsymbol{\mathcal{V}}}_i^h = \hat{\boldsymbol{J}}\dot{\boldsymbol{\theta}}_i^h \tag{8}$$

Next, we normalise the principal twists to obtain the principal screws:

$$\hat{\$}_i^h = \hat{\boldsymbol{\mathcal{V}}}_i^h/\mathrm{real}(\|\hat{\boldsymbol{\mathcal{V}}}_i^h\|_d) \tag{9}$$

It may be noted that after the transformation, $\hat{\boldsymbol{g}}$ is diagonalised, i.e., the dual inner-product of two non-identical principal screws vanish:

$$\left\langle \hat{\$}_i^h, \hat{\$}_j^h \right\rangle_d = 0 + \epsilon 0 \Leftrightarrow i \neq j \tag{10}$$

From the definition of inner product of screws (see, e.g., [13]) it follows immediately that the principal screws *meet at one point in space orthogonally.* The pitches along these screws, being the double of the generalised eigenvalues of $\boldsymbol{g}_0$ with respect to $\boldsymbol{g}$, are also extremal[1]. Therefore, the screws $\hat{\$}_i^h$, $i = 1, 2$ or $1, 2, 3$ are indeed the principal screws described by Ball [3] and Hunt [4]. We term the set of screws the h-basis of $\mathbb{P}^5$ [11]. This approach is novel to the best of our knowledge, and it seems to have the following advantages over existing work:

- The problem of identification of the principal screws is reduced to a generalised eigenproblem, which is well-studied in literature (see, e.g., [14]). Therefore, the properties of screw systems can be analysed using standard algebraic concepts.
- The conditions for a generic screw system to reduce to the special cases (as listed in [4]) can be obtained symbolically in closed-form, in terms of the coefficients of the generalised characteristic polynomial corresponding to Eq. (6).

[1] A more direct proof of this claim in Appendix A.

- For three-DOF rigid-body motion, sizes of $\boldsymbol{g}$ and $\boldsymbol{g}_0$ are limited to 3×3, and hence the generalised characteristic polynomial corresponding to Eq. (6) is restricted to a *cubic* at the most. Therefore, we can solve the problem in closed form. Further, since $\boldsymbol{g}$ is positive-definite in this case, we are also guaranteed to get real eigenvalues and eigenvectors (see, e.g., [14]).

We present the closed-form results for two-, and three-systems below. The 1-system is trivial, as its only constituent screw serves as the principal screw.

3 Analysis of Two-, Three-systems in the h-basis

In this section, we present the symbolic expressions for the principal screws of h-basis in terms of the input screw parameters. These exact expressions represent new contributions of this paper.

We use the notation c_θ, s_θ for $\cos\theta$ and $\sin\theta$ etc. The perpendicular distance between screws $\hat{\$}_i$ and $\hat{\$}_j$ is denoted by d_{ij} and the angle is denoted by α_{ij}. Further, c_i and s_i denote $\cos\theta_i$ and $\sin\theta_i$ respectively, and c_{ij}, s_{ij} denote $\cos\alpha_{ij}$ and $\sin\alpha_{ij}$ respectively.

3.1 Two-system

We first derive the closed-form expressions of the principal screws. We then recover the classical equation of the cylindroid associated with the general two-system using the dual algebra formulation. We also present a few special cases within the two-system.

3.1.1 Derivation of the h-basis

In this case, the input screws are denoted by $\hat{\$}_1$ of pitch h_1 and $\hat{\$}_2$ of pitch h_2. The generalised characteristic equation is a quadratic of the form:

$$a_0 h^2 + a_1 h + a_2 = 0 \qquad (11)$$

where $a_0 = 4s_{12}^2$, $a_1 = 4s_{12}(d_{12}c_{12} + (h_1 + h_2)s_{12})$ and $a_2 = \det(\boldsymbol{g}_0) = 4h_1h_2 - (h_1+h_2)^2c_{12}^2 - d_{12}^2 s_{12}^2 + d_{12}s_{12}(h_1+h_2)$. This equation may be solved to obtain the principal pitches as:

$$h_1^h = (h_1 + h_2) + d_{12}\frac{c_{12}}{s_{12}} - \frac{D}{s_{12}^2}$$
$$h_2^h = (h_1 + h_2) + d_{12}\frac{c_{12}}{s_{12}} + \frac{D}{s_{12}^2}, \quad s_{12} \neq 0$$

where $D = \sqrt{d_{12}^2 + (h_1 - h_2)^2}$. The (non-normalised) eigenvectors are given by:

$$\dot{\boldsymbol{\theta}}_1^h = (h_2 - h_1 + Ds_{12}, (h_1 - h_2)c_{12} + d_{12}s_{12})^T$$
$$\dot{\boldsymbol{\theta}}_2^h = (h_2 - h_1 - Ds_{12}, (h_1 - h_2)c_{12} + d_{12}s_{12})^T$$

The corresponding principal screws are obtained by mapping the eigenvectors by $\hat{\boldsymbol{J}}$ and normalising the result (as in equations (8,9)). The principal screws intersect orthogonally at a point in $\mathbb{R}^3$. We translate the origin to this point, and align the new coordinate axes along $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_1 \times \boldsymbol{q}_2$, where $\boldsymbol{q}_1, \boldsymbol{q}_2$ are parallel to $\hat{\$}_1^h, \hat{\$}_2^h$ respectively. In this new frame, principal screws attain the simple expressions $\hat{\$}_i^h = \boldsymbol{e}_i(1 + \epsilon h_i^h)$, $i = 1, 2$.

3.1.2 Derivation of the cylindroid

A general screw in a two-system can be expressed as a one-parameter system in the parameter $\theta \in [0, 2\pi]$:

$$\hat{\$} = \hat{\$}_1^h c_\theta + \hat{\$}_2^h s_\theta = c_\theta \boldsymbol{e}_1 + s_\theta \boldsymbol{e}_2 + \epsilon(h_1^h c_\theta \boldsymbol{e}_1 + h_2^h s_\theta \boldsymbol{e}_2)$$

Writing $\hat{\$}$ as $\boldsymbol{s} + \epsilon \boldsymbol{s}_0$, and noting that $\|\boldsymbol{s}\| = 1$, the foot of the perpendicular from the origin may be found as: $\boldsymbol{r}_0 = \boldsymbol{s} \times \boldsymbol{s}_0 = (0, 0, (h_2^h - h_1^h)s_\theta c_\theta)^T$. Denoting $(h_2^h - h_1^h)s_\theta c_\theta$ by z and writing $x = r\cos\theta, y = r\sin\theta$, $r \in \mathbb{R}^+$, we find:

$$z = (h_2^h - h_1^h)\frac{xy}{r^2} = (h_2^h - h_1^h)\frac{xy}{x^2 + y^2}$$

Rationalising, we arrive at the well-known equation of the cylindroid [3, 4]:

$$z(x^2 + y^2) + (h_1^h - h_2^h)xy = 0 \qquad (12)$$

3.1.3 Algebraic analysis of the special cases

One advantage of the algebraic formulation is that the conditions leading to special cases of screw systems can be obtained in terms of closed form algebraic equations.
Equal pitches (finite): The condition for the pitches to be the same is obtained by setting the discriminant of Eq. (11) to zero:

$$16(d_{12}^2 + (h_1 - h_2)^2)s_{12}^2 = 0 \qquad (13)$$

The last equation shows that the pitches are equal if either of the two conditions hold:

1. $s_{12} = 0$: Input screws are coaxial.
2. $d_{12} = 0 = h_1 - h_2$: The input screw axes intersect, and the screws have the same pitch.

However, condition 1 above implies that the screws have infinite pitch, since $a_0 = 4\det(\boldsymbol{g}) = 4s_{12}^2$, hence only condition 2 gives finite, equal pitches. This defines a one-parameter family of screws, the free parameter being the angle between the axes of the screws.
Infinite pitches: From the above expressions of h_1^h and h_2^h, both the pitches become infinite when $s_{12} = 0$, i.e., the screws are coaxial.

3.2 Three-system

We start with the closed-form results for the general three-system, and then derive the classical equation of the pitch-hyperboloid describing the distribution of pitches. The algebraic treatment of a few special cases described by Hunt [4] is also presented.

3.2.1 Derivation of the h-basis

The generalised characteristic Eq. (7) is a cubic in this case:

$$a_0h^3 + a_1h^2 + a_2h + a_3 = 0 \quad (14)$$

where $a_0 = -8\det(\boldsymbol{g})$, $a_3 = \det(\boldsymbol{g}_0)$. Expressions of the all the coefficients are obtained in closed form. However, we include only two of them here due to space constraint:

$$\begin{aligned} a_0 =& -8\det(\boldsymbol{g}) = 4(1 + A_1 - 4A_2) \\ a_1 =& -H(1 + A_1 - 4A_2) + 8((c_{12} - c_{23}c_{31})\, d_{12}s_{12} \\ &+ (c_{23} - c_{12}c_{31})\, d_{23}s_{23} + (c_{31} - c_{12}c_{23})\, d_{31}s_{31} \end{aligned}$$

where $A_1 = \cos 2\phi_{12} + \cos 2\phi_{23} + \cos 2\phi_{31}$, $A_2 = c_{12}c_{23}c_{31}$ and $H = h_1 + h_2 + h_3$. Eq. (14) admits only real solutions, as it arises from the simultaneous diagonalisation of two quadratic forms, $\boldsymbol{g}$ and $\boldsymbol{g}_0$, where $\boldsymbol{g}$ is positive definite (see, e.g., [14]). The *generalised eigenvectors* (non-normalised) corresponding to h_i^h may be obtained as:

$$\dot{\boldsymbol{\theta}}_i^h = (n_1, n_2, n_3)^T$$

$$\begin{aligned} n_1 =& 2c_{31}\left(h_i^h - h_2\right)\left(2h_i^h - h_1 - h_3\right) + 2d_{31}\left(h_i^h - h_2\right)s_{31} \\ &- \left(c_{12}\left(2h_i^h - h_1 - h_2\right) + d_{12}s_{12}\right) \\ &\times \left(c_{23}\left(2h_i^h - h_2 - h_3\right) + d_{23}s_{23}\right) \end{aligned}$$

$$\begin{aligned} n_2 =& 2c_{23}\left(h_i^h - h_1\right)\left(2h_i^h - h_2 - h_3\right) + 2d_{23}\left(h_i^h - h_1\right)s_{23} \\ &- \left(c_{12}\left(2h_i^h - h_1 - h_2\right) + d_{12}s_{12}\right) \\ &\times \left(c_{31}\left(2h_i^h - h_1 - h_3\right) + d_{31}s_{31}\right) \end{aligned}$$

$$\begin{aligned} n_3 =& c_{12}^2\left(h_1 + h_2\right)^2 + \left(d_{12}^2 + 4h_i^h\left(-h_i^h + h_1 + h_2\right)\right)s_{12}^2 \\ &- 4h_1h_2 - \sin\left(2\alpha_{12}\right)d_{12}\left(-2h_i^h + h_1 + h_2\right) \end{aligned}$$

The corresponding principal screws are obtained from equations (8,9).

3.2.2 Derivation of the pitch-hyperboloid

As in the case of two-system, we move the origin to the point of concurrence of the principal screws, and align the coordinate axes along the principal screw axes to obtain a new reference frame of $\mathbb{R}^3$, in which the principal screws have the form $\hat{\$}_i = \boldsymbol{e}_i(1 + \epsilon h_i^h)$, $i = 1, 2, 3$. Under the *unit speed* constraint, the possible screws can be written as a two-parameter family:

$$\begin{aligned} \hat{\$} &= \hat{\$}_1 l + \hat{\$}_2 m + \hat{\$}_3 n \quad (15) \\ &= l\boldsymbol{e}_1 + m\boldsymbol{e}_2 + n\boldsymbol{e}_3 + \epsilon(lh_1^h\boldsymbol{e}_1 + mh_2^h\boldsymbol{e}_2 + nh_3^h\boldsymbol{e}_3) \end{aligned}$$

where $l, m, n \in \mathbb{R}$, $l^2 + m^2 + n^2 = 1$. The foot of the perpendicular from the origin to the axis of $\hat{\$}$ may be found as:

$$\begin{aligned} \boldsymbol{r}_0 &= \boldsymbol{s} \times \boldsymbol{s}_0 \\ &= ((h_3^h - h_2^h)mn, (h_1^h - h_3^h)ln, (h_2^h - h_1^h)lm)^T \end{aligned}$$

By setting to zero l, m and n in turn, we obtain the cylindroids with the nodal axes along $\boldsymbol{e}_1, \boldsymbol{e}_2$, and $\boldsymbol{e}_3$ respectively. Therefore we arrive at the result obtained in[3, 4], that for an arbitrary three-DOF motion, the resultant screw axis lies on the intersection of three cylindroids concurrent at a point, with their nodal axes mutually orthogonal. We now derive the equation of the hyperboloid which describes the screw distribution completely [3, 4]. Let the screw axis pass through the point $(x, y, z)^T$. Hence its moment about the origin is given by $\boldsymbol{q}_0 = (x, y, z) \times \boldsymbol{q}$. However, from definitions, $\boldsymbol{q}_0 = \boldsymbol{s}_0 - h\boldsymbol{q}$. Therefore we have the equation:

$$(x, y, z)^T \times \boldsymbol{q} = \boldsymbol{s}_0 - h\boldsymbol{q}$$

Expanding this equation and rearranging, we obtain

$$\begin{pmatrix} h - h_1^h & -z & y \\ z & h - h_2^h & -x \\ -y & x & h - h_3^h \end{pmatrix}\begin{pmatrix} l \\ m \\ n \end{pmatrix} = 0 \quad (16)$$

For the above homogeneous equations to have a non-trivial solution, we must have the determinant of the matrix on the left hand side as zero. This condition yields

$$\begin{aligned} x^2(h - h_1^h) + y^2(h - h_2^h) + z^2(h - h_3^h) \\ + (h - h_1^h)(h - h_2^h)(h - h_3^h) = 0 \quad (17) \end{aligned}$$

The above equation gives the pitch h associated with a line passing through any arbitrary point (x, y, z), and corroborates with the results obtained by Hunt [4].

3.2.3 Algebraic analysis of the special cases

We now derive the conditions defining some of the special three-systems described by Hunt [4].

Two pitches are equal (all finite): We note that all the principal pitches are finite if $a_0 \neq 0$ in Eq. (14). To find the condition that the two of the principal pitches are the same, we scale the Eq. (14) by a_0 and compare with the equation

$$(x - \alpha)^2(x - \beta)^2 = 0 \quad (18)$$

where α, β represent the repeated, and non-repeated roots respectively. Equating the respective coefficients in equations (14,18), we obtain the following relationships between the coefficients:

$$\begin{aligned} 2\alpha + \beta + \frac{a_1}{a_0} &= 0 \\ \alpha^2\beta + \frac{a_3}{a_0} &= 0 \\ \alpha^2 + 2\alpha\beta - \frac{a_2}{a_0} &= 0 \end{aligned}$$

Eliminating α, β from the above equations, we obtain the required condition as:

$$4a_3a_1^3 - a_2^2a_1^2 - 18a_0a_2a_3a_1 + a_0\left(4a_2^3 + 27a_0a_3^2\right) = 0 \quad (19)$$

All pitches are equal (all finite): All the pitches are equal when the coefficients of the *standard form* of Eq. (14) vanish. In terms of the coefficients of the original equation, the required conditions are:

$$2a_1^3 - 9a_0a_1a_2 + 27a_0^2a_3^2 = 0 \quad (20)$$

$$a_1^2 - 3a_0a_2 = 0, \quad a_0 \neq 0 \quad (21)$$

One pitch infinite, two pitches unequal (finite): Two derive this set of conditions, we rewrite Eq. (14) in terms of $\sigma = 1/h$, to obtain:

$$a_3\sigma^3 + a_2\sigma^2 + a_1\sigma + a_0 = 0 \quad (22)$$

The condition for one of the pitches being infinite is equivalent to the vanishing of one of the σ's, i.e.,

$$a_0 = 0, \quad a_1, a_2, a_3 \text{ are not all zero} \quad (23)$$

Further, the two finite pitches are unequal, i.e., discriminant of the residual quadratic equation is nonzero:

$$a_2^2 - 4a_1a_3 \neq 0 \quad (24)$$

One pitch infinite, two pitches equal (finite): This case is similar to the last one, and the conditions are given by:

$$a_0 = 0$$

$$a_2^2 - 4a_1a_3 = 0, \quad a_1, a_2, a_3 \text{ are not all zero} \quad (25)$$

Two pitches infinite, one finite: From Eq. (22), the conditions for this case can be obtained as:

$$a_0 = a_1 = 0, \; a_2, a_3 \neq 0 \quad (26)$$

All pitches infinite: From Eq. (22), the conditions for this case can be obtained as:

$$a_0 = a_1 = a_2 = 0, \; a_3 \neq 0 \quad (27)$$

4 Analysis of Four-, Five-systems in the h-basis

We now study the cases where $\boldsymbol{g}$ is positive semi-definite. The formulation differs from the two-, and three-systems in that the generalised eigenproblem formulation is not possible in these cases, as these are characterised by $\det(\boldsymbol{g}) = 0$. Traditionally, in such cases the *reciprocal* screw systems are computed first, and their properties are analysed [3, 4]. However, we continue to use the same criteria for the identification of the principal screws- namely the extremal values of the pitch. Using the concept of degrees-of-freedom partitioning described in [11], we partition the space of input screws into two sets, having finite and infinite pitches respectively. We then find the principal screws of the former set. These screws are reciprocal to the principal screws of the reciprocal system by construction, and therefore we need not compute the reciprocal screw system explicitly as an intermediate step in our formulation. The basis can be completed by the addition of two *coaxial* screws of infinite pitches in accordance with the concept of classical *co-reciprocal* basis [4]. The steps involved for the analysis of four-system varies slightly from the five-system case, and we describe both of them below in details.

4.1 Four-systems

We study below the generic case considered in [3, 4]. The four-system considered constitutes of three screws of finite pitch and one of infinite pitch, signifying one pure translational DOF. The steps of computation may be delineated as follows.

- **Identify the pure translational twists:** We use the concept of *partitioning of degrees-of-freedom* described in [11] to find out the screw representing the pure translation. This is done by first obtaining the nullspace of $\boldsymbol{g}$, and then using it to obtain the screw $\hat{\$}_T$, on which all translational twists act:

$$\hat{\$}_T = \hat{\boldsymbol{J}}\dot{\boldsymbol{\theta}}^n, \qquad \boldsymbol{g}\dot{\boldsymbol{\theta}}^n = \boldsymbol{0} \quad (28)$$

  There are three linearly independent screws having finite pitches, which lie in the column-space of $\hat{\boldsymbol{J}}$. We denote these screws by $\hat{\$}_{Ci}, \; i = 1, 2, 3$, and their collection by $\hat{\$}_C$. Note that the original four-system is a direct sum of $\hat{\$}_T$ and $\hat{\$}_C$.

- **Decompose $\hat{\$}_C$ in sets of screws parallel and perpendicular to $\hat{\$}_T$:** It is known in literature, that the principal basis in this case consists of two coaxial screws of infinite pitch, and two screws of finite pitch orthogonal to this pair. In other words there is a two-parameter family of screws having finite pitch, which are orthogonal to the direction of pure translation. To find this set, we now decompose $\hat{\$}_C$ into two disjoint subspaces, one of which will have the axes parallel to the direction of $\hat{\$}_T$, and the other perpendicular to it. This is done by constructing a general screw, $\hat{\$}_c$, within the space $\hat{\$}_C$ by a linear combination of $\hat{\$}_{Ci}$:

$$\hat{\$}_c = c_i\hat{\$}_{Ci}, \; c_i \in \mathbb{R}, \; i = 1, 2, 3 \quad (29)$$

  We are interested in the solutions for c_i such that the following orthogonality condition is satisfied:

$$\text{real}\left(\left\langle \hat{\$}_\text{C}, \hat{\$}_\text{T} \right\rangle_\text{d}\right) = 0 \quad (30)$$

  Further, we introduce the normalization condition to confine $\hat{\$}_c$ to $\mathbb{P}^5$ (i.e., to ensure $\text{real}(\|\hat{\$}_c\|_d) = 1$):

$$c_1^2 + c_2^2 + c_3^2 = 1 \quad (31)$$

  Equations (30,31) define two constraints on c_i. There is a freedom in choosing one of the parameters, and without loss of generality, we choose c_3 arbitrarily. For each such choice of c_3, equations (30,31) yield two (generally distinct) pairs of solutions for the other interpolation coefficients c_1, c_2. These pairs lead to a pair of screws in $\hat{\$}_C$ via Eq. (29), which are perpendicular to $\hat{\$}_T$. We note that by the above construction, all the screws generated by linear combinations of these two screws are orthogonal to $\hat{\$}_T$, and they can have *only* finite pitch.

- **Find the principal screws of the two-screw system formed above:** Using the formulation for the two-system, we find the principal screws, $\hat{\$}_1^h$ and $\hat{\$}_2^h$, of the screw-system formed in the last step. Note that these two screws will be concurrent and mutually orthogonal, and they have the maximum and minimum pitch among the finite pitch screws of the original system. Therefore they form a part of the set of principal screws of the original system.
- **Complete the principal basis using the co-reciprocal convention:** The other two principal screws can have infinite pitch, hence by the above construction, they are restricted to be perpendicular to $\hat{\$}_1^h, \hat{\$}_2^h$. The location of these screws can be arbitrary. However, as per the definition of co-reciprocal basis [3], we choose these two coaxial screws to pass through the point of intersection of $\hat{\$}_1^h$ and $\hat{\$}_2^h$, with their axes parallel to that of $\hat{\$}_T$. The pitches of these two screws can be denoted as $\pm h_\gamma,\ h_\gamma \in [-\infty, \infty]$.
- **Position the origin of the principal basis at the center of the reciprocal cylindroid:** The two-screw system generated by the screws $\hat{\$}_1^h, \hat{\$}_2^h$ formed in the last step is not guaranteed to be reciprocal to the reciprocal of the original four-system, and the reciprocity theorem is not used in their construction. On the contrary, it may be shown that the h-basis formed thus is actually a two-parameter system, formed by the arbitrary translations of the cylindroid formed by $\hat{\$}_1^h, \hat{\$}_2^h$ in a plane perpendicular to the axis of $\hat{\$}_T$. This is possible since translations do not affect the pitch, or direction of the screws. The reciprocal basis of the original system is located at a particular point in this plane. This location, denoted by $\boldsymbol{O}^h(x, y, z)$, may be uniquely determined by using the fact that the two-system of reciprocal screws are reciprocal to the original four-system. The steps of the procedure are described below.

  1. Find the axes $\boldsymbol{q}_1^h, \boldsymbol{q}_2^h$ of the screws $\hat{\$}_1^h, \hat{\$}_2^h$ respectively. Denote their pitches by h_1^h, h_2^h respectively.
  2. Construct two screws, $\hat{\$}_{r1}$ and $\hat{\$}_{r2}$ coaxial with $\hat{\$}_1^h$ and $\hat{\$}_2^h$ respectively. Let their pitches be $(-h_1^h)$ and $(-h_2^h)$ respectively, and let them pass through the point $\boldsymbol{O}^h(x, y, z)$:
  $$\begin{aligned} \hat{\$}_{r1} &= \boldsymbol{q}_1^h + \epsilon(h_1^h \boldsymbol{q}_1^h + \boldsymbol{O}^h(x,y,z) \times \boldsymbol{q}_1^h) \\ \hat{\$}_{r2} &= \boldsymbol{q}_2^h + \epsilon(h_2^h \boldsymbol{q}_2^h + \boldsymbol{O}^h(x,y,z) \times \boldsymbol{q}_2^h) \quad (32) \end{aligned}$$
  3. Use the reciprocal product (denoted by $\langle \cdot, \cdot \rangle_r$) to generate three equations in x, y, z:
  $$\left\langle \hat{\$}_{ri}, \hat{\$}_j \right\rangle_r = 0 \quad (33)$$
  where $\hat{\$}_j$ is an element of the input four-system. Choose (i, j) such that the equations are distinct.
  4. Solve the above set of equations linearly for (x, y, z).

  Thus, in the process of locating the origin of the h-basis for the four-system uniquely, we locate the origin of the reciprocal two-system as well. The other screw parameters, such as the pitches, and the axes are obtained without using the reciprocal relationship. This final step only completes the determination of the reciprocal basis of the original four-system.

The above process is illustrated, and numerically verified for an example of a four-system in section 5.

4.2 Five-systems

The analysis of the five-system is not very different from the above, hence we describe it briefly below.

The starting point, once again, is the decomposition of the input screws into finite-, and infinite-pitch subsets. In this case, in addition to the three principal screws of finite pitch, there are two pure translation screws, i.e., all screws of infinite pitch are parallel to a *plane* in this case, as opposed to a line in the above case. We identify the screws of finite pitch as those perpendicular to this plane. Since there can be only one such direction, the finite pitch is also fixed along all the screws in that direction. Therefore all the screws perpendicular to the translation screws qualify as one element of the principal basis of the four-system. The other four principal screws of the h-basis are determined by constructing two pairs of co-reciprocal screws of indeterminate pitch. Finally, the reciprocal basis of the original screw system may be found by locating the single principal screw in a plane, as in the case of four-system. We note that the computations involved in the cases of four- and five-systems include solutions of linear and quadratic equations, in addition to the eigenproblem in the first case. However, since all the computations can be done symbolically, the final results can still be obtained in closed form.

We note that the general six-system associated with six-DOF rigid-body motion spans $\mathbb{P}^5$ completely, and as such it has no constraints [4]. The screws can have any pitch in $[\infty, \infty]$, and their axes can have any direction and location in $\mathbb{R}^3$. Therefore analysis in the h-basis can yield no information about such motions.

5 Illustrative Examples

In this section, we illustrate in closed-form some of the theoretical developments presented above with the example of a 3-R serial manipulator. We also demonstrate our algorithm for the four-system numerically.

5.1 Spatial 3-R manipulator

The *DH parameters* of the manipulator are given in Table 1.

Table 1: DH parameters of the spatial 3-R manipulator

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	d_1	θ_1
2	α_{12}	a_{12}	d_2	θ_2
3	α_{23}	a_{23}	d_3	θ_3

The matrix $\boldsymbol{g}$ is computed as:

$$\boldsymbol{g} = \begin{pmatrix} 1 & c_{12} & c_{12}c_{23} - c_2 s_{12} s_{23} \\ c_{12} & 1 & c_{23} \\ c_{12}c_{23} - c_2 s_{12} s_{23} & c_{23} & 1 \end{pmatrix}$$

Elements of $\boldsymbol{g}_0$ are obtained as:

$$\begin{aligned} &g_{011} = g_{022} = g_{033} = 0,\ g_{012} = g_{021} = -a_{12}s_{12} \\ &g_{013} = g_{031} = (d_2 s_2 s_{12} - (a_{23} + a_{12}c_2)c_{12})s_{23} \\ &\qquad\qquad - (a_{12} + a_{23}c_2)c_{23}s_{12} \\ &g_{023} = g_{032} = -a_{23}s_{23} \end{aligned}$$

The generalised characteristic Eq. (14) is written in terms of the architectural parameters and configuration $\boldsymbol{\theta}$ as:

$$\begin{aligned} &4s_2^2 s_{12}^2 s_{23}^2 h^3 - 4s_2 s_{12} s_{23}(c_2 d_2 s_{12} s_{23} + s_2(a_{23}c_{23}s_{12} \\ &+ a_{12}c_{12}s_{23}))h^2 + ((-a_{12}^2 - 2a_{23}c_2 s_{23}^2 a_{12} + (a_{12}^2 \\ &- a_{23}^2 c_2^2)c_{23}^2 - d_2^2 s_2^2 s_{23}^2 + 2a_{23}c_2 c_{23} d_2 s_2 s_{23})s_{12}^2 \\ &+ 2a_{12}c_2 c_{12} s_{23} s_{12}(d_2 s_2 s_{23} - a_{23}c_2 c_{23}) \\ &+ ((a_{23}^2 - a_{12}^2 c_2^2)c_{12}^2 - a_{23}^2)s_{23}^2 + \sin(2\alpha_{12})\sin(2\alpha_{23}) \\ &\times a_{12}a_{23})h + a_{12}a_{23}s_{12}s_{23}((a_{12} + a_{23}c_2)c_{23}s_{12} \\ &+ ((a_{23} + a_{12}c_2)c_{12} - d_2 s_2 s_{12})s_{23}) = 0 \end{aligned}$$

As expected, this is a cubic equation, which can be solved in closed-form using Cardano's formula (see, e.g., [15]). For the sake of illustration, we choose the architectural variables as $d_1 = 2$, $a_{12} = 1$, $\alpha_{12} = \pi/2$, $d_2 = 1/2$, $a_{23} = 1$, $\alpha_{23} = \pi/4$, $d_3 = 1/4$, $a_{34} = 1/4$, and the configuration variables as $\theta_1 = \pi/6$, $\theta_2 = \pi/4$, $\theta_3 = \pi/2$. For these numerical values, the symbolically computed principal pitches yield the following numerical values:

$$h_1^h = -0.987,\ h_2^h = 0.316,\ h_3^h = 2.171$$

The principal screws at this configuration, are given by:

$$\hat{\$}_1^h = (0.745, -0.451, 0.493)^T + \epsilon(0.940, 2.681, -0.972)^T$$
$$\hat{\$}_2^h = (0.428, -0.243, -0.870)^T + \epsilon(0.510, 1.565, -0.550)^T$$
$$\hat{\$}_3^h = (0.512, 0.859, 0.012)^T + \epsilon(-1.638, 3.503, 0.052)^T$$

It may be verified that the three principal screws intersect orthogonally at the point $\boldsymbol{O}^h(-0.308, -0.467, -3.207)^T$.

5.2 A general four-screw system

In this section we illustrate the procedure described in section 4 by deriving the h-basis of a randomly generated four-system. The input four-screw system consists of the following elements:

$$\hat{\$}_1 = (-0.815, 0.575, -0.073)^T - \epsilon(0.815, -0.575, 0.073)^T$$
$$\hat{\$}_2 = (-0.799, 0.330, 0.502)^T + \epsilon(-0.799, 0.330, 0.502)^T$$
$$\hat{\$}_3 = (-0.982, 0.142, -0.122)^T - \epsilon(0.982, -0.142, 0.122)^T$$
$$\hat{\$}_4 = (-0.992, -0.002, 0.129)^T - \epsilon(0.992, 0.002, -0.129)^T$$

The screw system is decomposed in finite, and infinite pitch components, and from infinite-pitch screw $\hat{\$}_T$, the translation direction is obtained as: $\boldsymbol{z} = (0.488, -0.810, -0.325)^T$. The three-system of screws with finite pitches are now decomposed in two sets, with axis parallel and perpendicular to this direction. We choose the free parameter c_3 in Eq. (31) as $c_3 = 1/3$. The two solutions for the interpolation coefficients c_i in Eq. (29) come out as:

$$c_1 = -0.182,\ c_2 = 0.925,\ c_3 = 1/3$$
$$c_1 = 0.050,\ c_2 = -0.942,\ c_3 = 1/3$$

Correspondingly, the two screws perpendicular to $\boldsymbol{z}$ are, respectively:

$$\hat{\$}_{C1} = -(0.435, 0.104, 0.393)^T - \epsilon(1.729, 0.759, -0.501)^T$$
$$\hat{\$}_{C2} = (0.119, -0.118, 0.474)^T + \epsilon(1.870, -0.440, 0.375)^T$$

The principal screws of the two-system generated by these two screws are:

$$\hat{\$}_1^h = (0.692, 0.586, -0.421)^T - \epsilon(1.238, -3.067, 2.266)^T$$
$$\hat{\$}_2^h = (0.532, -0.020, 0.847)^T + \epsilon(3.512, 0.279, -0.100)^T$$

The principal screws intersect at the point $\boldsymbol{p}(-1.016, -2.980, -2.210)^T$, and their pitches are given by $h_1^h = 1.893$, $h_2^h = 1.777$ respectively. The principal system of screws can be completed by adding to the pair $\hat{\$}_1^h, \hat{\$}_2^h$ two co-axial screws along the $\boldsymbol{z}$ axis, whose pitches are given by h_γ and $-h_\gamma$ respectively, where h_γ can take any value from $-\infty$ to ∞.

We also derive the reciprocal two-system using our procedure. Solving equations (33) in section 4, the origin of the reciprocal system is computed as:

$$\boldsymbol{O}^h(0.871, -2.972, -2.012)^T$$

The principal screws of the reciprocal system are computed from Eq. (32) as:

$$\hat{\$}_{r1} = (0.693, 0.586, -0.421)^T - \epsilon(3.741, -0.651, 0.751)^T$$
$$\hat{\$}_{r2} = (0.532, -0.020, 0.847)^T + \epsilon(1.611, 0.368, -3.102)^T$$

For the purpose of numerical verification, we also compute the reciprocal two-system following the procedure described in [4], and subsequently compute its principal screws using our method. The results are in numerical agreement up to the sense of these screws. Further, we generate a twist in the original four-system by random interpolation of the input screws: $\hat{\boldsymbol{\mathcal{V}}}_o = (-1.327, 0.680, 0.005)^T + \epsilon(-2.760, -3.152, 3.905)^T$. It may be verified that the reciprocal products, $\left\langle \hat{\boldsymbol{\mathcal{V}}}_o, \hat{\$}_{r1} \right\rangle_r$, $\left\langle \hat{\boldsymbol{\mathcal{V}}}_o, \hat{\$}_{r1} \right\rangle_r \sim \mathcal{O}(-16)$.

6 Conclusions

In this paper, we have presented an exact formulation for the derivation of the principal screws of a system of screws. We have shown that the complete solution of the dual eigenproblem in Eq. (2) leads to a generalised eigenproblem involving the real and dual parts of $\hat{\boldsymbol{g}}$. The eigenproblem is also shown to be solvable in closed form, and particular cases of screw systems have been studied symbolically.

The principal basis of screw systems derived solely from a new criterion, namely the extremisation of the pitch of the screws, is shown to be equivalent to the *principal screws* discussed in [3, 4]. We have derived the classical results of screw theory to demonstrate the consistency of our approach. However, the formulation presented here is novel, and the some of the closed-form results presented have been derived for the first time. The closed-form and numerical examples have been provided to illustrate the theory developed in this paper.

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Appendix

A The Extremality of h^h

From the definitions of $\boldsymbol{g}$, $\boldsymbol{g}_0$, we have

$$\dot{\boldsymbol{\theta}}^T \boldsymbol{g}_0 \dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}^T (\boldsymbol{J}_{\boldsymbol{\omega}}^T \boldsymbol{J}_{\boldsymbol{v}} + \boldsymbol{J}_{\boldsymbol{v}}^T \boldsymbol{J}_{\boldsymbol{\omega}}) \dot{\boldsymbol{\theta}} = \boldsymbol{\omega} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\omega}$$
$$\dot{\boldsymbol{\theta}}^T \boldsymbol{g} \dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}^T (\boldsymbol{J}_{\boldsymbol{\omega}}^T \boldsymbol{J}_{\boldsymbol{\omega}}) \dot{\boldsymbol{\theta}} = \boldsymbol{\omega} \cdot \boldsymbol{\omega} \quad (34)$$

Therefore, from the definition of pitch, we have: $h = \frac{\boldsymbol{\omega} \cdot \boldsymbol{v}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}$. Therefore $h = \frac{1}{2} \frac{\dot{\boldsymbol{\theta}}^T \boldsymbol{g}_0 \dot{\boldsymbol{\theta}}}{\dot{\boldsymbol{\theta}}^T \boldsymbol{g} \dot{\boldsymbol{\theta}}}$. To obtain the extremal values of pitch, we set $\frac{\partial h}{\partial \dot{\boldsymbol{\theta}}} = 0$. After a little manipulation, this leads to the condition:

$$(\boldsymbol{g}_0 - 2h\boldsymbol{g})\dot{\boldsymbol{\theta}} = \boldsymbol{0}, \ \det(\boldsymbol{g}) \neq 0 \quad (35)$$

Eq. (35) is identical with the eigenproblem in Eq. (6), and it is therefore established that the solution of the generalised eigenproblem gives rise to the extremal values of pitch.