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Some Classical Buckling Problems Revisited from a Continuum Approach

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Abstract

We reexamine some classical buckling problems using a prestress based three dimensional modal projection approach. The approach was recently developed for finding the critical speeds of arbitrary axisymmetric rotors. Both buckling and rotor whirl are problems of nonlinear elasticity, and the primary contribution of the approach is to intuitively identify key terms among all the nonlinear terms present that essentially determine the critical parameter of interest (load, for buckling; and speed, for whirl). Here, we briefly state the reduced formulation (that retains only key terms), and use modal projections to find the buckling loads for four systems. The primary contribution of the present paper lies in that the treatment of these classical and simple problems, though based on a continuum formulation, is *analytical*. The match obtained with known solutions to these problems suggests that the approach may be useful for a variety of other problems as well.

Keywords: Buckling, Prestress, Modal Projection

1 Introduction

In this paper we reexamine some classical buckling problems using a prestress based three dimensional modal projection approach. This approach was recently developed to find the critical speeds of arbitrary axisymmetric rotors using finite element analysis [1]. Both buckling and rotor whirl are, in a continuum setting, problems of nonlinear elasticity. A full treatment of these problems is possible using commercial codes, where several different nonlinear terms are handled in a uniform setting. Our aim in [1], as well as here, is to intuitively identify the key terms involved which, with conceptually simple modal projections, give good results. For verifying that the primary effects being sought are indeed captured by a few terms, in this paper we implement the same continuum formulation *analytically*, and consider some classical buckling problems.

Buckling is in some ways similar to rotor whirl at a critical speed. Both represent critical parameters (load and speed, respectively) where new solutions appear and uniqueness is lost. Both problems involve a state of prestress that is in equilibrium in the nominal state, but contributes a small restoring or destabilizing term under a small deflection, which acts in addition to the usual elastic restoring forces. We have found in [1] that due accounting of this effect of the prestress gives, numerically at least, excellent approximations to the actual critical speeds as computed in other, more complete and/or reliable, ways. In this paper, we apply the same simplifying idea in analytical treatments of some classical buckling problems.

We study four problems. The first is the buckling of a slender column under axial compression; the second is the buckling of a slender vertical column under gravity loading or self-weight; the third is the buckling of a beam on an elastic foundation, subjected to axial compression; and the fourth and final problem is the buckling of a circular ring subjected to external radially inward loading (external pressure). For all four problems, analytical solutions are available in the classical literature [2]. We will obtain the same results from a completely different approach. As indicated above, the main point of this work is not that we *get* these known results, but that we get them using a simplified but intuitive modal projection based continuum approach.

We acknowledge that genuinely nonlinear treatments of buckling problems sometimes give results that are far from those obtained using simplified, linearized, “classical” analyses. However, such differences are not of interest here; here we are interested in clarifying the minimal terms from nonlinear elasticity that yield those same classical results. We also mention that the formulation we use to begin our modal projection seems new in rotordynamics, but is not new in continuum treatments of nonlinear elasticity (see, e.g., [4], chapter 5).

2 A Modal Projection Method

In this section we derive a modal projection method for calculating the critical buckling load. We start with the governing equations of equilibrium, project them onto one or more lateral bending modes, and use the principle of virtual work. The key step in this three dimensional approach is to include the effects of the stress produced by the loading that eventually causes buckling. We begin with the general equilibrium equation [3]

$$\nabla \cdot (\mathbf{FS}) = 0. \tag{1}$$

Here $\mathbf{F}$ is the deformation gradient and $\mathbf{S}$ is the second Piola-Kirchhoff stress at the material point of interest.

We split the total displacement in the buckled state into two components: the axial displacement and the lateral bending,

$$\mathbf{u} = \varepsilon \mathbf{u}_0 + a\phi \;, \tag{2}$$

where ε and a are bookkeeping coefficients and ϕ is a lateral bending mode of the structure under consideration. The critical buckling load is that at which infinitesimal bending solutions are possible in addition to the unbent original configuration. Thus, uniqueness of the solution is lost at the buckling load. This will be used in the calculations below.

Assuming the St. Venant-Kirchhoff stress strain relation [3], the second Piola-Kirchhoff stress is written as

$$\mathbf{S} = \lambda (\text{tr}\, \mathbf{E})\, \mathbf{I} + 2\mu \mathbf{E},$$

where λ and μ are Lame constants and $\mathbf{E}$ is the Green strain tensor given by

$$\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T + \nabla\mathbf{u}^T\nabla\mathbf{u}\right).$$

However, as also discussed in [1], we note that the key nonlinear physical effect that contributes to buckling is that of an infinitesimal disturbance (bending) of a pre-existing significant stress state (due to the applied load). This disturbance is accounted for by $\mathbf{F}$ in Eq. (1). Strain terms that are nonlinear in the displacement, in our opinion, play an insignificant role; and so $\mathbf{S}$ in Eq. (1) is here approximated using linear terms only. This approximation is to be analytically tested for some buckling problems, in what follows.

Interestingly, the dropped strain terms nonlinear in the displacements turn out to be *identical* to terms representing the effect of load induced configuration changes, which we will also drop in Eq. (4) below. In this sense, our use of the St. Venant-Kirchhoff constitutive relation is notional. Note also that *if* these dropped nonlinear terms were in fact important, then their presence would presumably be needed in the usual formulations of buckling; their absence there supports our approach. Observe, finally, that our arguments for dropping these terms remain intuitive; and for this reason, analytical examination of some familiar problems provides an element of support that a rigorous "full" treatment might not need.

Proceeding, we take $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$ where

$$\mathbf{E}_0 = \frac{\varepsilon}{2}\left(\nabla\mathbf{u}_0 + \nabla\mathbf{u}_0^T\right), \text{ and } \mathbf{E}_1 = \frac{a}{2}\left(\nabla\phi + \nabla\phi^T\right).$$

We can then split $\mathbf{S}$, the second Piola-Kirchhoff stress, into prestress and bending components. The prestress component is given by

$$\mathbf{S}_0 = \lambda (\text{tr}\, \mathbf{E}_0)\, \mathbf{I} + 2\mu \mathbf{E}_0,$$

the bending component is given by

$$\mathbf{S}_1 = \lambda (\text{tr}\, \mathbf{E}_1)\, \mathbf{I} + 2\mu \mathbf{E}_1,$$

and

$$\mathbf{S} = \mathbf{S}_0 + \mathbf{S}_1. \tag{3}$$

We now turn to the deformation gradient

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u} = \mathbf{I} + \varepsilon\nabla\mathbf{u}_0 + a\nabla\phi.$$

We again note (or intuitively hypothesize) that the key term of interest involves the bending-induced disturbance of the load-induced stress state $\mathbf{S}_0$. This, consistent with neglect of load-induced changes in geometry, lets us ignore $\mathbf{u}_0$ and write, as an acceptable approximation,

$$\mathbf{F} = \mathbf{I} + a\nabla\phi. \tag{4}$$

Thus finally at $O(a)$ we have

$$\nabla\cdot(\nabla\phi\mathbf{S}_0) + \nabla\cdot\mathbf{S}_1 = 0.$$

A similiar equation starting with prestress due to applied load is also derived in [4] (Chapter 5, Eq. V13) for calculating the critical load for loss of elastic stability.

3 Virtual Work

Considering a virtual displacement $\delta\mathbf{w}$, we have

$$\int_V (\nabla\cdot(\mathbf{FS}))\cdot\delta\mathbf{w}\, dV = 0. \tag{5}$$

The critical buckling load is that at which the uniqueness of solution is lost. This happens when the cofficient of a in the above equation becomes zero. Substituting Eqs. (3) and (4) into the left hand side, and retaining only terms linear in a, we get

$$\int_V (\nabla\cdot(\nabla\phi\mathbf{S}_0))\cdot\delta\mathbf{w}\, dV + \int_V (\nabla\cdot\mathbf{S}_1)\cdot\delta\mathbf{w}\, dV = 0. \tag{6}$$

This equation can be solved to obtain the critical buckling load. A similar equation, *via* similarly dropped terms, but including centrifugal body forces, was used in [1] for rotor whirl.

4 Buckling of Columns

We now obtain the critical buckling load of Euler-Bernoulli columns subjected to three different boundary conditions, as shown in figure (1), using the present prestress based method. We start with Eq. (6), written now as

$$\int_V (\nabla\cdot\nabla\phi\mathbf{S}_0)\cdot\delta\mathbf{w}\, dV + \int_V (\nabla\cdot\mathbf{S}_1)\cdot\delta\mathbf{w}\, dV = 0, \tag{7}$$

where $\delta\mathbf{w}$ is a virtual displacement. Let the displacement of the column, bending in the X-Z plane, be ϕ. Under Euler-Bernoulli assumptions

$$\phi = \begin{bmatrix} u \\ 0 \\ -x\dfrac{du}{dz} \end{bmatrix}, \tag{8}$$

where u is the displacement in the X direction. Ignoring Poisson's effects the displacement in the Y direction is taken

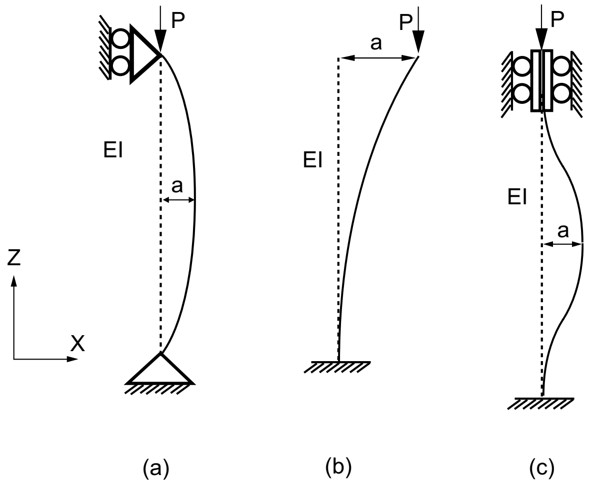

Figure 1: Buckling of columns: Case(a) Pinned-pinned (b) fixed-free (c) fixed-fixed.

as zero. The prestress $\mathbf{S}_0$ in the column arises from the axial load P and the components of this stress are

$$\mathbf{S}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P/A \end{bmatrix}, \tag{9}$$

where A is the cross sectional area. The stress $\mathbf{S}_1$ is due to infinitesimal bending and the components of this stress are given by

$$\mathbf{S}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -Ex\dfrac{d^2u}{dz^2} \end{bmatrix}, \tag{10}$$

where E is the Young's modulus of the material. Substituting these values in Eq. (7) and taking the virtual displacement as

$$\delta\mathbf{w} = \begin{bmatrix} \delta w \\ 0 \\ -x\dfrac{d(\delta w)}{dz} \end{bmatrix}, \tag{11}$$

we get

$$\int_V \left(-\frac{P}{A}\frac{d^2u}{dz^2}\cdot\delta w + \left(E - \frac{P}{A}\right)x^2\frac{d^3u}{dz^3}\cdot\frac{d(\delta w)}{dz}\right)dV = 0.$$

Or

$$\int_0^l\int_{\bar{A}}\left(-\frac{P}{A}\frac{d^2u}{dz^2}\cdot\delta w + \left(E - \frac{P}{A}\right)x^2\frac{d^3u}{dz^3}\cdot\frac{d(\delta w)}{dz}\right)d\bar{A}\,dz = 0.$$

where $\bar{A}$ represents the cross sectional area as a domain of integration, distinct from A, which we use to denote the numerical value of the total cross sectional area. Since none of the variables u, z and w vary across the cross section, the above integral becomes

$$\int_0^l \left(-P\frac{d^2u}{dz^2}\cdot\delta w + \left(E - \frac{P}{A}\right)I\frac{d^3u}{dz^3}\cdot\frac{d(\delta w)}{dz}\right)\,dz = 0,$$

where I is the area moment of inertia of the cross section. Integrating by parts the second term in the above integrand, imposing boundary conditions, and noting specifically that for case (b) in figure 1 we require $\left.\dfrac{d^3u}{dz^3}\right|_l = 0$, we obtain with no further restrictions:

$$\int_0^l \left(-P\frac{d^2u}{dz^2} + \left(\frac{P}{A} - E\right)I\frac{d^4u}{dz^4}\right)\delta w\,dz = 0. \tag{12}$$

Now since δw is arbitrary, the term in the brackets in the above integrand must be identically zero, giving the governing equation for buckling of a beam subjected to an axial compressive load P. Since the Young's modulus $E \gg P/A$ for the problems of interest[1], the above equation reduces to

$$EI\frac{d^4u}{dz^4} + P\frac{d^2u}{dz^2} = 0,$$

which is the familiar equation governing buckling of Euler-Bernoulli beams. We emphasize that the above equation can be obtained using the classical strength of materials approach; the interesting thing here is merely that, starting from nonlinear elasticity, a continuum formulation, and an intuitive interpretation that lets us simplify the continuum formulation, we have in fact obtained the same equation. Similar calculations, implemented numerically, have enabled us to find accurately the critical speeds of several axisymmetric rotors in [1].

Although we were able to obtain the governing differential equation for buckling of Euler-Bernoulli columns, our interest in this paper (as in [1]) is in modal projections. Accordingly, we will use modal projections to solve the problem, i.e., we will assume a functional form for the displacement function $u = af(z)$, set the virtual displacement $\delta w = \delta a\, f(z)$, and use Eq. (12) to obtain the buckling load. We consider three sets of boundary conditions.

(a) A pinned-pinned beam.

For this case, shown in figure (1a), we take

$$u = a\sin\left(\frac{\pi z}{l}\right),$$

where l is the length of the beam and a is the maximum displacement occuring at the center of the beam. Substituing this into Eq. (12) and letting $\delta w = \delta a \sin\left(\frac{\pi z}{l}\right)$, integrating and solving for P we get

$$P_{cr} = \frac{\pi^2 EI}{l^2}.$$

This matches the classical result.

(b) A cantilevered column.

For this case, shown in figure (1b), we let

$$u = a\left(1 - \cos\left(\frac{\pi z}{2l}\right)\right),$$

[1]In all subsequent calculations in this article, P/A has consistently been dropped in comparison with E.

which satisfies the essential boundary conditions at the fixed end. Now a is the displacement of the free end. Proceeding similarly, we obtain

$$P_{cr} = \frac{\pi^2 EI}{4l^2},$$

matching the classical result.

(c) A fixed-fixed column.

For this case, shown in figure (1c), we let

$$u = a\left(1 - \cos\left(\frac{2\pi z}{l}\right)\right).$$

This satisfies the essential boundary conditions at the fixed ends. Now a is the displacement of the center. Again proceeding as before, we obtain

$$P_{cr} = \frac{4\pi^2 EI}{l^2},$$

matching the classical result.

5 Columns with Other Loading

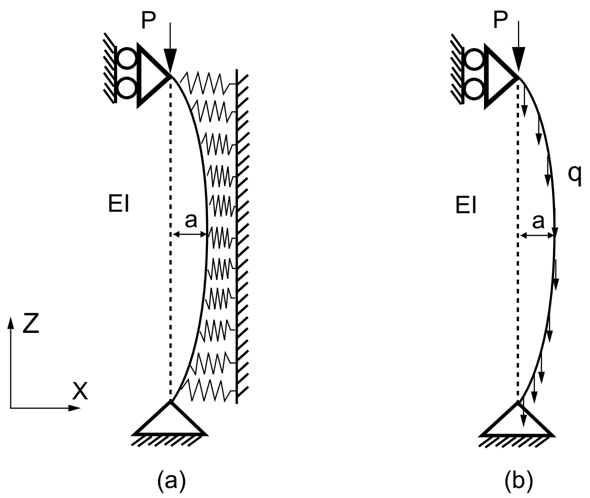

Figure 2: (a) Buckling of a pinned-pinned column on an elastic foundation. (b) Buckling of a pinned-pinned column under its own weight.

Consider a pinned-pinned beam supported by an elastic foundation (as in [2]) of stiffness k, as shown in figure (2a), and subjected to an axial compressive force P. In this case, in addition to the axial stresses due to infinitesimal bending, there will be compressive stresses due to elastic forces from the side. Instead of calculating this stress we can directly use the work done by the elastic foundation through the virtual displacement. Thus Eq. (6) changes to

$$\int_V (\nabla \cdot (\nabla\phi \mathbf{S}_0)) \cdot \delta\mathbf{w}\, dV + \int_V (\nabla \cdot \mathbf{S}_1) \cdot \delta\mathbf{w}\, dV - \int_0^l ku\,\mathbf{i} \cdot \delta\mathbf{w}\, dz = 0, \quad (13)$$

where $\mathbf{i}$ is the unit vector along the X direction. The thickness of the beam cross section is accounted for in k. A derivation of the above equation from Eq. (6) is given in appendix (C).

Assuming the displacement along the X direction to be

$$u = a\sin\left(\frac{\pi z}{l}\right),$$

ϕ is obtained from Eq. (8). The virtual displacement is taken as $\delta\mathbf{w} = (\delta a/a)\phi$. Substituting $\delta\mathbf{w}$, ϕ, Eq. (9) and Eq. (10) into Eq. (13), the critical buckling load P is obtained as

$$P_{cr} = \frac{\pi^2 EI}{l^2}\left(1 + \frac{kl^4}{EI\pi^4}\right).$$

This again matches the classical result (see [2], equation 2-37, with $m = 1$).

5.1 A simply supported column under axial load and self weight

Let the column have a mass density of q/g per unit length, where g is the acceleration due to gravity; in other words, the self weight per unit length is q. In this section we derive the critical value of axial load P for a given q (see figure (2b)). Including the effects of axial load as well as self weight, the prestress $\mathbf{S}_0$ in this case is given by

$$\mathbf{S}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{P}{A} + \frac{q}{A}(z-l) \end{bmatrix}.$$

Using this, Eq. (11) and taking $\mathbf{S}_1$ and ϕ from Eq. (8) and Eq. (10) respectively and substituting in Eq. (6) we get the following equation

$$\int_0^l \left(EI\frac{d^3u}{dz^3}\right) \cdot \frac{d(\delta w)}{dz} - \left(P\frac{d^2u}{dz^2} - q(z-l)\frac{d^2u}{dz^2} - q\frac{du}{dz}\right)\delta w\, dz = 0. \quad (14)$$

Now, for easy comparison with the results in [2], we take $q = \frac{\pi^2 EI}{l^3}$ and calculate P_{cr}, for comparison with the classical result,

$$P_{cr} = \frac{4.77EI}{l^2}.$$

Using as a first approximation $u = a\sin\left(\frac{\pi z}{l}\right)$, we use Eq. (14) and obtain

$$P_{cr} = \frac{4.93EI}{l^2}.$$

The small mismatch is due to the fact that the actual buckled shape does not concide with our assumed u. We can improve the accuracy by taking two terms,

$$u = a\sin\left(\frac{\pi z}{l}\right) + b\sin\left(\frac{2\pi z}{l}\right).$$

In this case we obtain two equations by letting δw in Eq. (14) to be $\delta a \sin\left(\frac{\pi z}{l}\right)$ and $\delta b \sin\left(\frac{\pi z}{l}\right)$ respectively. The critical load is obtained by setting the determinant of the resulting matrix of coefficients to zero so that buckling solutions are possible. The matrix obtained is

$$\begin{bmatrix} \frac{\pi^2(18Pl^2+9ql^3-18EI\pi^2)}{36l^3} & \frac{20q}{9} \\ \frac{20q}{9} & \frac{\pi^2(18Pl^2+9ql^3-72EI\pi^2)}{9l^3} \end{bmatrix}$$

Substituting $q = \frac{\pi^2 EI}{l^3}$ and setting the determinant of the above matrix to zero, we calculate the critical load as

$$P_{cr} = \frac{4.77EI}{l^2},$$

which matches the classical result to the number of significant digits shown. Alternatively, by setting $P = 0$ we can calculate the critical q at which the column will buckle as

$$q_{cr} = \frac{1.88EI}{l^3},$$

which again matches the classical result (an interpolation is required between discrete values given in table 2.8 of [2]).

6 Buckling of a Ring

In this section we derive the critical buckling load for a thin ring subjected to a uniform external radial loading (or external "pressure"; see figure (3). The classical solution is [2, 5]

$$q = \frac{3EI}{R^3},$$

where E is the Young's Modulus of the material, I is the moment of inertia of the cross section and R is the radius of the ring.

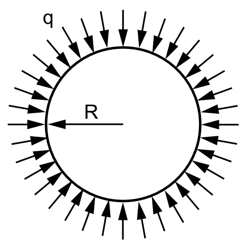

Figure 3: A uniformly loaded thin ring.

We start as usual with Eq. (6).

$$\int_V (\nabla\cdot\nabla\phi\mathbf{S}_0)\cdot\delta\mathbf{w}\,dV + \int_V (\nabla\cdot\mathbf{S}_1)\cdot\delta\mathbf{w}\,dV = 0, \qquad (15)$$

where ϕ is the displacement of a point on the ring, $\mathbf{S}_0$ is the prestress due to the uniform pressure loading, $\mathbf{S}_1$ is the stress due to infinitesimal bending from the original configuration and $\delta\mathbf{u}$ is the virtual displacement.

The displacement of the ring contains a radial component w as shown on the right of figure (4). Cross sections rotate, and there is a tangential displacement as well (to preserve inextensibility along the neutral axis). No displacement is taken in the z direction (Poisson's ratio $\nu = 0$). The radial displacement of the ring is taken as $w = a\cos(2\theta)$. Using this, the displacement ϕ of a point on the ring in a cylindrical coordinate system is given by

$$\phi = a\cos(2\theta)\,\hat{\mathbf{e}}_r - 2a\left(\frac{r}{R}-1\right)\sin(2\theta)\,\hat{\mathbf{e}}_\theta,$$

where $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ are unit vectors along the radial and tangential directions respectively. Rewriting using matrix notation, we have

$$\phi = \begin{bmatrix} a\cos(2\theta) \\ -2a(\frac{r}{R}-1)\sin(2\theta) \\ 0 \end{bmatrix}.$$

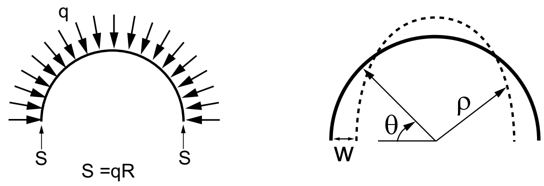

Figure 4: Force and displacement.

Next we calculate the prestress $\mathbf{S}_0$ due to the uniformly applied pressure q. Due to the uniform pressure q a compressive force S develops in the ring. This force is assumed to be uniform along the cross section. From a free body diagram of half the ring (see figure (4) left) the magnitude of the force S can be calculated as

$$S = qR.$$

Thus the prestress expressed in cylindrical coordinates is

$$\mathbf{S}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{qR}{A} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where A is the cross sectional area of the ring and the negative sign indicates compressive stress.

Next, we calculate the stress induced due to the infinitesimal bending of the ring. We consider a small fiber along the neutral axis of the ring of length $Rd\theta$. After bending the radius of curvature changes to ρ and the subtended angle changes to $d\psi$. Since the fiber along the neutral axis does not change in length

$$Rd\theta = \rho d\psi.$$

For a typical fiber away from the neutral axis, the change in length can be calculated as follows. Let the fiber be at a distance $y = r - R$ from the neutral axis. Then its original length is

$$L_0 = (R + y)d\theta,$$

and the length after bending is

$$L = (\rho + y)d\psi.$$

The elongation strain in the fiber is

$$\varepsilon_\theta = \frac{L - L_0}{L_0} = y\left(\frac{1}{\rho} - \frac{1}{R}\right).$$

The tangential stress therefore is

$$\sigma_\theta = E\varepsilon_\theta = Ey\left(\frac{1}{\rho} - \frac{1}{R}\right).$$

Using the well known relation between change in curvature and radial displacement w (see [2]) and using $y = r - R$ we get

$$\sigma_\theta = E(r - R)\left(\frac{1}{R^2}\frac{d^2w}{d\theta^2} + \frac{w}{R^2}\right).$$

We will use ϕ corresponding to $w = a\cos(2\theta)$ as indicated above. The bending stress components in cylindrical coordinates is

$$\mathbf{S}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The virtual displacement is taken as

$$\delta\mathbf{w} = \begin{bmatrix} \delta a\cos(2\theta) \\ -2\delta a(\frac{r}{R} - 1)\sin(2\theta) \\ 0 \end{bmatrix}.$$

Substituting the above expressions for $\mathbf{S}_1$ and $\mathbf{S}_0$ and ϕ along with the formulas for calculating gradient of vectors and divergence of a second order tensor in cylindrical coordinates as given in Appendix (A) into Eq. (15) we get

$$\int_0^{2\pi}\int_{R-c_1}^{R+c_2} g\,(R, q, E, \theta, r)\; r\,dr\,d\theta = 0, \qquad (16)$$

where the function g is given in the appendix (B). c_1 and c_2 are the distances of the extreme fibers from the neutral axis. Upon performing the integration and assuming $R \gg c_1, c_2$ we get for the critical load

$$q_{cr} = \frac{3EI}{R^3},$$

matching the classical result.

7 Conclusions

In this paper we have presented an alternate way of deriving the critical loads for some classical buckling problems. The method is general and can in principle be applied to solve many buckling problems with a suitable choice of modes (and possibly numerical evaluation of integrals, as in [1]).

However, the main contribution of this paper is *not* in providing an easier or more efficient way to solve a buckling problem. Rather it is to illustrate that starting with the nonlinear elastic continuum equations, *identifying* the key contributing terms, and performing modal projections, one can usefully solve buckling problems. The paper also shows the similarity between the buckling problems studied here and the rotor whirl problems studied in [1]. Both are nonlinear elasticity problems in which a state of prestress is disturbed from equilibrium and contributes a restoring force. This insight, perhaps newer to the rotor dynamicist than to the buckling analyst, is a key contribution of this paper.

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Appendix

A Grad and Div in Cylindrical Coordinates

Let ϕ be a vector with cylindrical components ϕ_r, ϕ_θ and ϕ_z. Then

$$\nabla\phi = \begin{bmatrix} \dfrac{\partial\phi_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial\phi_r}{\partial\theta} - \phi_\theta\right) & \dfrac{\partial\phi_r}{\partial z} \\ \dfrac{\partial\phi_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial\phi_\theta}{\partial\theta} + \phi_r\right) & \dfrac{\partial\phi_\theta}{\partial z} \\ \dfrac{\partial\phi_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial\phi_z}{\partial\theta} & \dfrac{\partial\phi_z}{\partial z} \end{bmatrix}.$$

For a second order tensor T,

$$\nabla\cdot\mathbf{T} = \left\{ \begin{array}{c} \dfrac{\partial T_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial T_{r\theta}}{\partial\theta} + \dfrac{T_{rr} - T_{\theta\theta}}{r} + \dfrac{\partial T_{rz}}{\partial z} \\ \dfrac{\partial T_{\theta r}}{\partial r} + \dfrac{1}{r}\dfrac{\partial T_{\theta\theta}}{\partial\theta} + \dfrac{T_{r\theta} + T_{\theta r}}{r} + \dfrac{\partial T_{\theta z}}{\partial z} \\ \dfrac{\partial T_{zr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial T_{z\theta}}{\partial\theta} + \dfrac{T_{zr}}{r} + \dfrac{\partial T_{zz}}{\partial z} \end{array} \right\}.$$

B The Function g in Eq. (16)

$$g(R,q,E,\theta,r) = \frac{1}{2rAR^3}(40qR^3r\cos(4\theta) - 24qR^3r$$
$$-15qR^4\cos(4\theta) + 9qR^4 - 27Er^2AR\cos(4\theta)$$
$$+21Er^2AR + 15ErAR^2\cos(4\theta) - 9ErAR^2$$
$$+20qR^2r^2 - 20qR^2r^2\cos(4\theta) - 12Er^3A + 12Er^3A\cos(4\theta)).$$

C Derivation of Eq. (13)

We begin with Eq. (6)

$$\int_V (\nabla\cdot(\nabla\phi\mathbf{S}_0))\cdot\delta\mathbf{w}\,dV + \int_V (\nabla\cdot\mathbf{S}_1)\cdot\delta\mathbf{w}\,dV = 0.$$

Here $\mathbf{S}_1$ is the stress induced in the column due to infintesimal bending. For a column on an elastic foundation, in addition to the usual bending stresses given by Eq. (10), there arises a normal traction on the face contacting the elastic foundation. Thus the stress can be split into two parts as

$$\mathbf{S}_1 = \mathbf{S}_b + \mathbf{S}_f,$$

where $\mathbf{S}_b$ is the bending induced part given by Eq. (10) and $\mathbf{S}_f$ is the stress due to the force from the foundation. The $\mathbf{S}_f$ contribution to the second term in Eq. (6) is

$$\int_V (\nabla\cdot\mathbf{S}_f)\cdot\delta\mathbf{w}\,dV.$$

Since $\mathbf{S}_f$ is symmetric, the following is an identity (see [3]):

$$\int_V (\nabla\cdot\mathbf{S}_f)\cdot\delta\mathbf{w}\,dV = \int_V \nabla\cdot(\mathbf{S}_f\cdot\delta\mathbf{w})\,dV - \int_V \mathbf{S}_f{:}\nabla(\delta\mathbf{w})\,dV.$$

Taking $\delta\mathbf{w} = (\delta a/a)\phi$ following discussions in section (4), $\mathbf{S}_f$ and $\nabla(\delta\mathbf{w})$ are given by

$$\mathbf{S}_f = \begin{bmatrix} \frac{ql}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\nabla(\delta\mathbf{w}) = \frac{\delta a}{a}\begin{bmatrix} 0 & 0 & \frac{du}{dz} \\ 0 & 0 & 0 \\ -\frac{du}{dz} & 0 & -x\frac{d^2u}{dz^2} \end{bmatrix},$$

whence $\mathbf{S}_f{:}\nabla(\delta\mathbf{w}) = 0$.

Using the divergence theorem the other term becomes

$$\int_V \nabla\cdot(\mathbf{S}_f\cdot\delta\mathbf{w})\,dV = \int_S (\mathbf{S}_f\cdot\delta\mathbf{w})\cdot\mathbf{n}\,dS = \int_S (\mathbf{S}_f\cdot\mathbf{n})\cdot\delta\mathbf{w}\,dS,$$

where $\mathbf{n}$ is the normal to the surface S of the column which rests on the elastic foundation (note that, for our chosen coordinate system, $\mathbf{n} = \mathbf{i}$). Now $\mathbf{S}_f\cdot\mathbf{n} = \mathbf{t}$, the traction force on the surface S, which is equal to the force due to compression of the elastic foundation and hence $\mathbf{t} = -ku\mathbf{i}/W$, where W is the width of the contacting face (assuming, for simplicity, a rectangular cross section). Hence,

$$\int_V (\nabla\cdot\mathbf{S}_f)\cdot\delta\mathbf{w}\,dV = -\int_S k\frac{u}{W}\mathbf{i}\cdot\delta\mathbf{w}\,dS, = -\int_0^l ku\,\mathbf{i}\cdot\delta\mathbf{w}\,dz,$$

since thickness is accounted for in stiffness k. Adding this term to the other terms of Eq. (6) we get Eq. (13), the governing equation for buckling of a column on elastic foundation.