Instantaneous Mobility and Constraint of a 3-PPS Parallel Mechanism

Jing-Shan Zhao*, Fulei Chu, Li-Ping Wang and Zhi-Jing Feng
Department of Precision Instruments and Mechanology, Tsinghua University, Beijing 100084, P. R. China.

* Corresponding author (email: jingshanzhao@mail.tsinghua.edu.cn.)

Abstract

This paper investigates the mobility and constraint of a spatial parallel mechanism whose end-effector has three degrees of freedom. The novelty of the mechanism is that its end-effector should output two rotations and one translation with 3-PPS kinematic chains. In this paper, the degree of freedom of an end-effector of a mechanism and the number of actuations required to control the end-effector are defined, individually. And then, the exponential product procedure to analyze the twists and wrenches of a kinematic chain is deduced in one fixed Cartesian coordinate system. The mobility and constraint of the end-effector are addressed from the viewpoint of instantaneous free motions and the number of actuations needed to control it. Actuation scheme analysis indicates that the degree of freedom of an end-effector and the number of independent actuations needed to uniquely control the end-effector should not be ambiguously represented by one concept such as the general mobility of a mechanism.

Keywords: Mobility, Constraint, Spatial Mechanism, End-effector, and Kinematic Chain

1 Introduction

The mobility and constraint of a mechanism are of primary importance in the theory of mechanisms, and have been extensively discussed either in analytical methods or in quick calculation approaches based on algebra summations of the number of the links, joints and the constraints induced by the joints in the previous literature during the past 150 years [1]. This lengthy development process on the one hand demonstrates the complexity and difficulty of this problem, and on the other hand indicates that it is of vital importance both in theory and in engineering applications.

As to the development of the mobility of a mechanism, Gogu [1] made a relatively detailed review. Therefore, this paper does not intend to make a full development about the history of mobility of a mechanism. Generally, the restrictions of the method based on pure algebra summations of the numbers of the links, joints and the constraints induced by the joints have been discussed widely in [2-7]. Rico and Ravani [6,7], Rico, Gallardo and Ravani [8], Rico and et al [9] made contemporary investigations on the mobility analysis of parallel mechanisms and kinematic chains with Lie group and Lie algebra. In applications, it is very convenient to investigate the constraint and mobility of a mechanism with reciprocal screw theory. Phillips and Hunt [10], Waldron [2], Hunt [3], Ohwovoriole and Roth [11], Sugimoto and Duffy [12][13], Gibson and Hunt [14][15], Phillips [4][16], McCarthy [17], Rico and Duffy [18-20], Tsai and Lee [21], Murray, Li and Sastry [22], Zhang and Xu [23], Fang and Huang [24], Huang and Wang [25], Bandyopadhyay and Ghosal [26] et al made great contributions to the applications of screw theory in mechanisms after Ball [27]. Traditionally, the mobility of a mechanism is thought of as the number of independent parameters to define the configuration of a mechanism.

However, with the advent of spatial parallel mechanisms, the primary considerations of the designers have been focused on nothing but the mobility of the end-effector and its controllability. Consequently, it is urgently necessary to discriminate between the degree of freedom of an end-effector and the number of actuations to uniquely control the end-effector under a configuration. Therefore, this paper focuses on the two aspects of the mobility of a mechanism. For convenience, it first introduces a pair of definitions:

Definition 1: The degree of freedom (DOF) of an end-effector totally characterizes the motions of the end-effector including the number, type and direction of the independent motions.

Definition 2: The configuration degree of freedom (CDOF) of a mechanism with a prescribed end-effector indicates the number of actuations required to uniquely control the end-effector under a configuration.

Obviously, the DOF of an end-effector in number is not larger than 6 but the number of actuations required to uniquely control the end-effector might be any nonnegative integer. Bearing the above two definitions in mind, one can fall into two steps to investigate the mobility of a mechanism—the DOF of the end-effector and the CDOF of the mechanism with a specified end-effector. The former definition indicates the full instantaneous mobility properties of the end-effector while the later one tells us the instantaneous controllability of the mechanism system.

2 Architecture of the Mechanism and the Mobility and Constraint of Its Kinematic Chains

The mechanism proposed in this paper is shown in figure 1. The end-effector $E_1E_2E_3$ of the mechanism has three identical *PPS* kinematic chains $P_iA_iE_i$ (i=1,2,3) connected by the fixed base $B_1B_2B_3$. It has two rotational DOFs and one translational DOF (2R1T). The subtended angles between any two guides of the fixed base are 120º.

It is very convenient to express the kinematics of a serial chain by exponential product of the revolute angles of the joints in the chain [22]. Therefore, the free motions of each *PPS* kinematic chain might be well depicted by an exponential product formula. Consequently, this section will investigate the mobility of a *PPS* kinematic chain with reciprocal screw theory [3-5][16][27] based on exponential product [22].

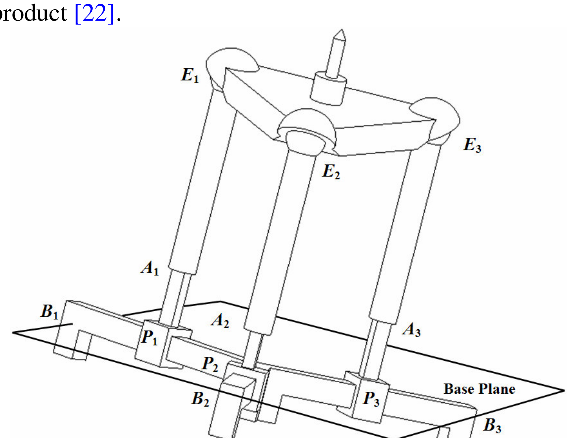

Figure 1: the Architecture of a Spatial Parallel Mechanism with 2R1T End-Effector

A *PPS* kinematic chain of the end-effector $E_1E_2E_3$, $P_1A_1E_1$ for instance, and the fixed Cartesian coordinate system are shown in figure 2. For the sake of building up the mathematic model, a Cartesian coordinate system is set here by letting x-axis be superimposed with the guide line of the prismatic joint P_1, y-axis locate in the base plane $B_1B_2B_3$, and z-axis be perpendicular to the base plane $B_1B_2B_3$ and directing to the end-effector $E_1E_2E_3$.

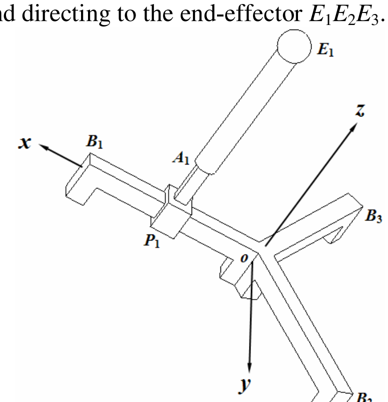

Figure 2: a Series *PPS* Kinematic Chain Connected with the Base

Assume that $SO(3)=\{R\in\Re^{3\times3}|RR^T=I,\det(R)=1\}$ denotes a 3×3 special orthogonal matrix group where I denotes an identical matrix. Group $SO(3)$ represents the three dimensional rotations in a Cartesian coordinate space, and therefore, it is also alternatively called configuration space.

The exponential transformation for a point on a link rotated θ about a joint axis $\boldsymbol{\omega}$ can be expressed with:

$$\mathbf{R}=\exp\left(\hat{\omega}\theta\right) \tag{1}$$

where $\wedge$ denotes a wedge operation such that the following equation holds for a vector $\boldsymbol{\omega}$:

$$\hat{\omega}=\left(\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}\right)^{\wedge}=\begin{bmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{bmatrix} \tag{2}$$

and θ denotes the rotational angle about the axis ω.

Let $so(3)=\{\mathbf{S}\in\Re^{3\times3}|\mathbf{S}^T=-\mathbf{S}\}$, if $\hat{\omega}\in so(3)$ and $\theta\in R$, there must be $\exp\left(\hat{\omega}\theta\right)\in SO(3)$. It is not difficult to prove that the exponential mapping from $so(3)$ to $SO(3)$ is surjective.

Define groups:

$se(3)=\left\{\left(\hat{\omega},\boldsymbol{v}\right)\middle|\hat{\omega}\in so(3),\boldsymbol{v}\in\Re^3\right\}$, and

$SE(3)=\{(\mathbf{R},\boldsymbol{p})|\mathbf{R}\in SO(3),\boldsymbol{p}\in\Re^3\}=\Re^3\times SO(3)$.

Obviously, the element $(\mathbf{R},\boldsymbol{p})\in SE(3)$ can be used to express the transformation from one coordinate system to another.

Suppose the coordinates of a twist are denoted by $\xi=\begin{bmatrix}\omega\\ v\end{bmatrix}$. Obviously, if a Vee operation, $\vee$, and a wedge operation, $\wedge$, are also defined such that $\hat{\xi}=\begin{bmatrix}\hat{\omega} & v\\ 0 & 0\end{bmatrix}$ and $\left(\hat{\xi}\right)^{\vee}=\begin{bmatrix}\hat{\omega} & v\\ 0 & 0\end{bmatrix}^{\vee}=\begin{bmatrix}\omega\\ v\end{bmatrix}=\xi$, then $e^{\hat{\xi}\theta}\in SE(3)$.

If one assumes $g(0)$ to represent the initial transformation of a rigid body relative to the Cartesian coordinate system, then the final configuration, still with respect to the same Cartesian coordinate system, is given by:

$$g(\theta)=\exp\left(\hat{\xi}\theta\right)g(0) \tag{3}$$

Thus, the exponential map for a twist gives the relative motion of a rigid body. Therefore, the exponential of a twist is a mapping from initial configuration to the final one. It can be proved that given $g \in SE(3)$, there exists $\hat{\xi} \in se(3)$ and $\theta \in R$ such that $g = \exp\left(\hat{\xi}\theta\right)$. The forward kinematics map for a series kinematic chain is given by:

$$g(\theta) = \prod_{i=1}^{n} \exp\left(\hat{\xi}_i \theta_i\right) g(0) \tag{4}$$

where $g(\theta)$ represents the rigid motion of the end-effector connected by the series kinematic chain, ξ_i represents the ith associated twist in the chain, and θ_i represents the ith relative angle.

Equation (4) can be expanded to

$$\mathbf{g}(\theta) = \begin{bmatrix} \mathbf{R}(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix} \tag{5}$$

The velocity of the end-effector can be specified by defining wedge and Vee operations in the absolute coordinate system:

$$\hat{v} = \dot{\mathbf{g}}\mathbf{g}^{-1} = \begin{bmatrix} \dot{\mathbf{R}}\mathbf{R}^T & -\dot{\mathbf{R}}\mathbf{R}^T p + \dot{p} \\ 0 & 0 \end{bmatrix} \tag{6}$$

$$v = \left(\hat{v}\right)^{\vee} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \left(\dot{\mathbf{R}}\mathbf{R}^T\right)^{\vee} \\ -\dot{\mathbf{R}}\mathbf{R}^T p + \dot{p} \end{bmatrix} \tag{7}$$

Therefore, with equations (4), (6) and (7), one obtains the velocity of the end-effector. From equation (6), one might find that:

$$\hat{v} = \dot{g}\, g^{-1} = \left(\sum_{i=1}^{n} \frac{\partial g}{\partial \theta_i} \dot{\theta}_i\right) g^{-1} = \sum_{i=1}^{n} \left(\frac{\partial g}{\partial \theta_i} g^{-1}\right) \dot{\theta}_i \tag{8}$$

In twist coordinates, equation (8) can be written as:

$$v = \mathbf{J}_t \dot{\theta} \tag{9}$$

where

$$\mathbf{J}_t = \left[\left(\frac{\partial g}{\partial \theta_1} g^{-1}\right)^{\vee} \quad \left(\frac{\partial g}{\partial \theta_2} g^{-1}\right)^{\vee} \quad \Lambda \quad \left(\frac{\partial g}{\partial \theta_n} g^{-1}\right)^{\vee} \right]$$

and $\mathbf{J}_t$ represents the Jacobian matrix of the kinematic chain in the absolute Cartesian coordinate system, and $\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \Lambda & \dot{\theta}_n \end{bmatrix}^T$ and $\dot{\theta}$ indicates the magnitude vector of general rotational velocities of the chain.

Each column of the Jacobian matrix $\mathbf{J}_t$ represents the twist coordinates of the corresponding joint in the Cartesian coordinate system. Therefore, the twist coordinates of the point attached to the rigid end-effector are given with equation (8) or (9). Given a fixed Cartesian coordinate system, one can directly write out the Jacobian matrix $\mathbf{J}_t$ with the above conclusion. For the mechanism shown in figure 1, the Jacobian matrix of the kinematic chain $P_1A_1E_1$, denoted by $\mathbf{J}_1$, can be obtained in the fixed Cartesian coordinate system shown in figure 2 in accordance with the following steps.

Step 1: write out the twist of the kinematic pair in the chain, individually.

The twist of the prismatic pair P_1 can be directly obtained:

$$\$_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T \tag{10}$$

The twist of the prismatic pair A_1 can be directly obtained:

$$\$_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T \tag{11}$$

For the spherical kinematic pair E_1, if its Cartesian coordinates are denoted by $r_E = \begin{pmatrix} x_{E_1} & y_{E_1} & z_{E_1} \end{pmatrix}^T$, the twists for the three orthogonal axes can be expressed as:

$$\$_{E_{1i}} = \begin{bmatrix} e_i \\ r_E \times e_i \end{bmatrix} \tag{12}$$

where $\mathbf{e}_i (i = 1,2,3)$ indicates the three orthogonal directions of the spherical joint E_1.

Consequently, according to equation (12), the three twists of the spherical joint can be expressed as:

$$\begin{cases} \$_{E_{11}} = \begin{pmatrix} 1 & 0 & 0 & 0 & z_{E_1} & -y_{E_1} \end{pmatrix}^T \\ \$_{E_{12}} = \begin{pmatrix} 0 & 1 & 0 & -z_{E_1} & 0 & x_{E_1} \end{pmatrix}^T \\ \$_{E_{13}} = \begin{pmatrix} 0 & 1 & 0 & y_{E_1} & -x_{E_1} & 0 \end{pmatrix}^T \end{cases} \tag{13}$$

Step 2: write out the Jacobian matrix of the kinematic chain by letting the twists of the chain be the columns one by one.

For example, the Jacobian matrix of the kinematic chain $P_1A_1E_1$ can be obtained from equations (10), (11) and (13):

$$\mathbf{J}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -z_{E_1} & y_{E_1} \\ 0 & 0 & z_{E_1} & 0 & -x_{E_1} \\ 0 & 1 & -y_{E_1} & x_{E_1} & 0 \end{bmatrix} \tag{14}$$

The twist of the terminal point, E_1, of the kinematic chain $P_1A_1E_1$ is:

$$v_{E_1} = \mathbf{J}_1 c_1 \tag{15}$$

where c_1 represents a vector whose elements are the relative rotational or translational speeds of the corresponding joints and $c_1 = (\omega_{P_1} \quad \omega_{A_1} \quad \omega_{E_{11}} \quad \omega_{E_{12}} \quad \omega_{E_{13}})^T$ here.

For the sake of length of the paper, the exponential product theory of screw is not fully developed here. Please refer to [22] for the details. The reciprocal screw theory indicates that the constraints exerted to the point attached to the rigid end-effector are reciprocal to the twist of the point. Therefore,

$$\boldsymbol{\tau}^T E \boldsymbol{v} = \left(\boldsymbol{\tau}^T E \mathbf{J}_t\right)\dot{\boldsymbol{\theta}} = 0 \tag{16}$$

where $\boldsymbol{\tau}$ indicate the wrench coordinates in the absolute coordinate system, the superscript T indicates the transpose of a vector or matrix, and $E = \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Because of the independency and arbitrariness of $\dot{\theta}$, the necessary and sufficient criteria for equation (16) can be simplified as:

$$\left(\mathbf{J}_t\right)^T E \boldsymbol{\tau} = 0 \tag{17}$$

Vice versa, the twist(s) of a rigid body can be obtained with the following formula if the wrenches exerted to it are known:

$$\boldsymbol{\tau}^T E \mathbf{J}_t = 0 \tag{18}$$

Consequently, it is very convenient to investigate the instantaneous mobility and constraint of a series kinematic chain with reciprocal screw theory.

With equation (17), one immediately obtains:

$$\boldsymbol{\tau}_1 = F_1\begin{pmatrix} 0 & 1 & 0 & -z_{E_1} & 0 & x_{E_1} \end{pmatrix}^T \tag{19}$$

where F_1 indicates the magnitude of the wrench.

The physical meaning of τ_1 is that the terminal point, E_1, of kinematic chain $P_1A_1E_1$ is subjected to a constraint force passing through it and with a direction of $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$. In the like manner, one can get the twists and wrenches of the points E_2 and E_3 in the same Cartesian coordinate system shown in figure 2, respectively:

$$\mathbf{J}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{2} & 0 & 0 & -z_{E_2} & y_{E_2} \\ \frac{\sqrt{3}}{2} & 0 & z_{E_2} & 0 & -x_{E_2} \\ 0 & 1 & -y_{E_2} & x_{E_2} & 0 \end{bmatrix} \tag{20}$$

where $\begin{pmatrix} x_{E_2} & y_{E_2} & z_{E_2} \end{pmatrix}$ represent the Cartesian coordinates of the spherical joint E_2, and

$$\boldsymbol{v}_{E_2} = \mathbf{J}_2 \boldsymbol{c}_2 \tag{21}$$

where $\boldsymbol{c}_2 = \begin{pmatrix} \omega_{P_2} & \omega_{A_2} & \omega_{E_{21}} & \omega_{E_{22}} & \omega_{E_{23}} \end{pmatrix}^T$.

$$\boldsymbol{\tau}_2 = F_2\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{1}{2}z_{E_2} & \frac{\sqrt{3}}{2}z_{E_2} & \frac{1}{2}x_{E_2} - \frac{\sqrt{3}}{2}y_{E_2} \end{pmatrix}^T \tag{22}$$

where F_2 indicates the magnitude of the wrench.

$$\mathbf{J}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{2} & 0 & 0 & -z_{E_3} & y_{E_3} \\ -\frac{\sqrt{3}}{2} & 0 & z_{E_3} & 0 & -x_{E_3} \\ 0 & 1 & -y_{E_3} & x_{E_3} & 0 \end{bmatrix} \tag{23}$$

where $\begin{pmatrix} x_{E_3} & y_{E_3} & z_{E_3} \end{pmatrix}$ represent the Cartesian coordinates of the spherical joint E_3, and

$$\boldsymbol{v}_{E_3} = \mathbf{J}_3 \boldsymbol{c}_3 \tag{24}$$

where $\boldsymbol{c}_3 = \begin{pmatrix} \omega_{P_3} & \omega_{A_3} & \omega_{E_{31}} & \omega_{E_{32}} & \omega_{E_{33}} \end{pmatrix}^T$.

$$\boldsymbol{\tau}_3 = F_3\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & \frac{1}{2}z_{E_3} & \frac{\sqrt{3}}{2}z_{E_3} & -\frac{1}{2}x_{E_3} - \frac{\sqrt{3}}{2}y_{E_3} \end{pmatrix}^T \tag{25}$$

where F_3 indicates the magnitude of the wrench.

3 The Mobility of the End-Effector and the Actuators Required to Control It

The terminal constraints of the three kinematic chains have been obtained in section 2, which are also the constraints exerted to the end-effector $E_1E_2E_3$ by the three chains. Therefore, the constraints applied to $E_1E_2E_3$ are directly expressed by equations (19), (22) and (25):

$$\tau = \mathbf{J}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \tag{26}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 & -z_{E_1} & 0 & x_{E_1} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{1}{2}z_{E_2} & \frac{\sqrt{3}}{2}z_{E_2} & \frac{1}{2}x_{E_2} - \frac{\sqrt{3}}{2}y_{E_2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & \frac{1}{2}z_{E_3} & \frac{\sqrt{3}}{2}z_{E_3} & -\frac{1}{2}x_{E_3} - \frac{\sqrt{3}}{2}y_{E_3} \end{bmatrix}^T.$$

With equation (16), one can also deduce that

$$\left(\mathbf{J}\right)^T E \boldsymbol{v} = 0 \tag{27}$$

where $\boldsymbol{v}$ denote the twists of the end-effector.

Immediately, one obtains

$$\boldsymbol{v} = \mathbf{J}_{\mathrm{v}} \boldsymbol{K} \tag{28}$$

where $\mathbf{J}_{\mathrm{v}}$ denotes the kinematic Jacobian matrix of the end-effector and

$$\mathbf{J}_{\mathrm{v}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & \dfrac{2z_1 - z_2 - z_3}{2x_1 - x_2 - x_3 + \sqrt{3}(y_2 - y_3)} & \dfrac{\sqrt{3}\begin{bmatrix} \sqrt{3}(y_3z_1 + y_2z_1 - y_3z_2 - y_2z_3) \\ + x_1z_2 + x_2z_3 + x_3z_1 - x_1z_3 - x_3z_2 - x_2z_1 \end{bmatrix}}{3\left[2x_1 - x_2 - x_3 + \sqrt{3}(y_2 - y_3)\right]} & \dfrac{\begin{bmatrix} \sqrt{3}(y_2z_1 - y_3z_1) \\ + x_1z_3 + x_1z_2 - x_2z_1 - x_3z_1 \end{bmatrix}}{2x_1 - x_2 - x_3 + \sqrt{3}(y_2 - y_3)} & 0 \\ 0 & 1 & \dfrac{\sqrt{3}(z_2 - z_3)}{2x_1 - x_2 - x_3 + \sqrt{3}(y_2 - y_3)} & \dfrac{-x_1z_2 + x_3z_2 + \sqrt{3}y_3z_2 - x_1z_3 + x_2z_3 - \sqrt{3}y_2z_3}{2x_1 - x_2 - x_3 + \sqrt{3}(y_2 - y_3)} & \dfrac{-\sqrt{3}x_1(z_2 - z_3)}{2x_1 - x_2 - x_3 + \sqrt{3}(y_2 - y_3)} & 0 \end{bmatrix}^T \tag{29}$$

$$\boldsymbol{K} = \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}^T \tag{30}$$

and $t_i (i = 1,2,3)$ denote any real values.

Therefore, the instantaneous mobility of the end-effector is expressed by equation (28). For any prescribed $\boldsymbol{K}$, the end-effector has one determinate instantaneous free motion represented with equation (28).

The rank of the twist matrix (29) represents the number of degrees of freedom of the end-effector. What must be noted is that equation (28) dynamically expresses the instant motion of the end-effector and consequently, the rank of the twist matrix (29) also dynamically expresses the instant DOF of the end-effector. In general configuration, there is

$$Rank(\mathbf{J}_{\mathrm{v}}) = 3 \tag{31}$$

It is not difficult to find from equation (28) that the free motion of the end-effector $E_1E_2E_3$ is a proper screw whose pitch is a nonzero value in general, which is shown in figure 3.

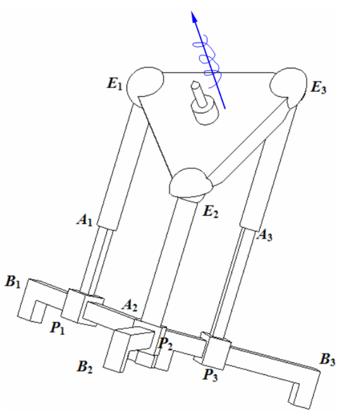

Figure 3: the General Free Motion of the End-Effector

The direction of the instantaneous twist of the end-effector might be any vector but z-axis provided that $2x_1 - x_2 - x_3 + \sqrt{3}(y_2 - y_3) \neq 0$. When $z_1 = z_2 = z_3 = z_E$, the kinematic Jacobian of the end-effector will be simplified as:

$$\mathbf{J}_{\mathrm{v}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & z_E & 0 \\ 0 & 1 & 0 & -z_E & 0 & 0 \end{bmatrix}^T \tag{32}$$

The instantaneous free motion of the end-effector $E_1E_2E_3$ might be represented by a twist:

$$\boldsymbol{v} = \begin{pmatrix} t_2 & t_3 & 0 & -z_Et_3 & z_Et_2 & t_1 \end{pmatrix}^T \tag{33}$$

Equation (33) indicates that the end-effector $E_1E_2E_3$ might rotate about any axis within the plane $z = z_E$ provided that $z_1 = z_2 = z_3 = z_E$. Therefore, equation (28) can be utilized to control the end-effector to output two decoupled rotations perpendicular to z-axis and one translation along z-axis.

After obtaining the mobility of the end-effector, another question might arise——Can three actuators completely control the end-effector $E_1E_2E_3$? This is quite another problem.

As is proposed in [28], the controllability of the end-effector should be discussed separately after obtaining the instantaneous mobility of the end-effector. According to the above analysis, three actuations must be given to the mechanism. In the mechanism shown in figure 1, the feasible actuators might be assigned to any three of the 6 prismatic joints. Therefore, there are $C_6^3 = 20$ possible actuation schemes for the mechanism in total, which can be grouped into 6 different schemes in essence. If the actuation scheme is named after the joints assigned as actuators, these six different scheme can be individually denoted by $P_1P_2P_3$, $A_1A_2A_3$, $P_1P_2A_{1(2)}$, $P_1P_2A_3$, $P_1A_2A_3$, $P_{2(3)}A_2A_3$ actuation schemes. In the following section, these six different actuation schemes will be investigated, individually.

Scheme 1: $P_1P_2P_3$ **Actuation Scheme**

Assign 3 actuators to the prismatic joints P_1, P_2, P_3.

If a set of 3 actuations are applied to these three sliders, the twist matrixes of the three kinematic chains, $P_1A_1E_1$, $P_2A_2E_2$ and $P_3A_3E_3$ can be obtained from equations (14), (20) and (23) by expelling the twist corresponding to the prismatic joints P_1, P_2, P_3:

$$\mathbf{J}_{S_1 1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -z_{E_1} & y_{E_1} \\ 0 & z_{E_1} & 0 & -x_{E_1} \\ 1 & -y_{E_1} & x_{E_1} & 0 \end{bmatrix} \tag{34}$$

$$\mathbf{J}_{S_1 2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -z_{E_2} & y_{E_2} \\ 0 & z_{E_2} & 0 & -x_{E_2} \\ 1 & -y_{E_2} & x_{E_2} & 0 \end{bmatrix} \tag{35}$$

$$\mathbf{J}_{S_13} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -z_{E_3} & y_{E_3} \\ 0 & z_{E_3} & 0 & -x_{E_3} \\ 1 & -y_{E_3} & x_{E_3} & 0 \end{bmatrix} \tag{36}$$

The terminal constraints of these three actuated kinematic chains can be similarly obtained according to equation (17), respectively:

$$\begin{cases} \boldsymbol{\tau}_{11} = F_{11}\begin{pmatrix}1 & 0 & 0 & 0 & z_{E_1} & -y_{E_1}\end{pmatrix}^T \\ \boldsymbol{\tau}_{12} = F_{12}\begin{pmatrix}0 & 1 & 0 & -z_{E_1} & 0 & x_{E_1}\end{pmatrix}^T \end{cases} \tag{37}$$

$$\begin{cases} \boldsymbol{\tau}_{21} = F_{21}\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2}z_{E_2} & \frac{1}{2}z_{E_2} & -\frac{\sqrt{3}}{2}x_{E_2}-\frac{1}{2}y_{E_2}\end{pmatrix}^T \\ \boldsymbol{\tau}_{22} = F_{22}\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{1}{2}z_{E_2} & \frac{\sqrt{3}}{2}z_{E_2} & \frac{1}{2}x_{E_2}-\frac{\sqrt{3}}{2}y_{E_2}\end{pmatrix}^T \end{cases} \tag{38}$$

$$\begin{cases} \tau_{31} = F_{31}\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2}z_{E_3} & \frac{1}{2}z_{E_3} & \frac{\sqrt{3}}{2}x_{E_3}-\frac{1}{2}y_{E_3}\end{pmatrix}^T \\ \tau_{32} = F_{32}\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & \frac{1}{2}z_{E_3} & \frac{\sqrt{3}}{2}z_{E_3} & -\frac{1}{2}x_{E_3}-\frac{\sqrt{3}}{2}y_{E_3}\end{pmatrix}^T \end{cases} \tag{39}$$

Therefore, the terminal constraints exerted to the end-effector can be obtained:

$$\tau = \begin{bmatrix}\tau_{11} & \tau_{12} & \tau_{21} & \tau_{22} & \tau_{31} & \tau_{32}\end{bmatrix} \tag{40}$$

Substituting equation (40) into equation (18) yields:

$$\boldsymbol{v} = k\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}^T \tag{41}$$

where k represents the magnitude of the twist and k might be any real value.

Equation (41) indicates that the end-effector will always have one free translation along z-direction under the three actuations in scheme 1 under any configurations. Therefore, the actuation scheme 1 can not safeguard the controllability of the end-effector even the same number of actuations as the DOFs of the end-effector are exerted to the mechanism. So, for this actuation scheme, one more actuation has to be applied to the mechanism. Suppose the additional actuation is exerted to the prismatic joint A_1, of course, A_2 or A_3 is also available for this additional actuator. It is not difficult to find that under these 4 actuations the mechanism shown in figure 1 is stable.

Alternatively, however, one can also select another actuation scheme, namely,

Scheme 2: $A_1A_2A_3$ **Actuation Scheme**

Exert three actuations to the prismatic joints A_1, A_2, and A_3. For this actuation scheme, it is not difficult to find that the end-effector does be stably controlled. And therefore, the number of actuators needed to control the end-effector is 3 for this actuation scheme.

Scheme 3: $P_1P_2A_{1(2)}$ **Actuation Scheme**

Exert 2 actuations to the prismatic joints P_1 and P_2, and the third actuation is assigned to prismatic joint A_1 or A_2. For this actuation scheme, it is not difficult to find that the end-effector does be stably controlled. And therefore, the number of actuators needed to control the end-effector is 3 for this actuation scheme.

With a similar analytical process, one can find that among the rest three actuation schemes, $P_1P_2A_3$, $P_1A_2A_3$ and $P_{2(3)}A_2A_3$, only $P_1A_2A_3$ and $P_{2(3)}A_2A_3$ can control the end-effector.

All in all, the mechanism proposed here demonstrates that the number of actuations required to control the end-effector of a same specified mechanism might be different if the actuation schemes are allowed to be different; on the other hand, it also indicates that the mobility of the end-effector and the actuations needed to control it should be treated differently.

4 Conclusions

The mobility of a mechanism is very important in the mechanism and machine theory. The DOF of an end-effector and the number of actuations required to control the motion of the end-effector are different and should not be depicted by one concept. Therefore, by proposing a novel spatial parallel mechanism whose end-effector could produce three free motions, including two decoupled rotations and one perpendicular translation, this paper investigates the analytical model to validly investigate the DOF of an end-effector and the number of actuations required to control the movement of the end-effector. With the actuation scheme analysis, it reveals that the concept of general mobility cannot effectively represent the end-effector's free motions and controllability, and therefore the DOF of an end-effector and the number of actuations required to control it should be disposed separately.

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